TRANSVERSE INSTABILITY OF THE LINE SOLITARY WATER-WAVES

FREDERIC ROUSSET AND NIKOLAY TZVETKOV

Abstract. We prove the linear and nonlinear instability of the line solitary water waves with respect to transverse perturbations.

Contents

1. Introduction

The water waves problem that is to say the study of fluid motions in the presence of a free surface has been the object of many studies in the past thirty years. This problem which is highly nonlinear is very interesting in many aspects. The study of the well-posedness of the Cauchy problem has been widely studied recently [36, 24, 4, 9, 25, 11, 34]. A lot of progress has also been made in the rigorous justification of important asymptotic models like the KdV or KP equations [12, 3, 33]. Finally, let us also mention that the study of bifurcation of travelling waves [6, 8, 22, 19] or more complicated patterns [10, 23, 30] has also attracted a lot of attention. An important aspect of the theory which is in some sense in the intersection of the above aspects is the study of the dynamical stability of important particular patterns like solitary waves.

Our goal here is to study the stability of the line solitary waves constructed by Amick and Kirchgässner in [6]. In the context of the water waves model, in the physical situation, the velocity of the fluid depends on three variables, while the free surface of the fluid which is also unknown is two-dimensional. We shall refer to this situation as the two-dimensional case. We shall call one-dimensional waves or line waves solutions of the water waves equations for which the surface of the fluid is invariant by translation in one direction.

The stability of the line solitary waves when submitted to one-dimensional perturbations was studied by Mielke in [26] where a conditional orbital stability result was established. This means that, as long as the solution of the water waves equations issued from a small one-dimensional perturbation of a line solitary wave exists in the energy space, it remains close to translates of the solitary wave in the energy norm. The precise statement is given below.

Here, we shall prove the nonlinear Lyapounov instability of these waves when they are submitted to perturbations depending in a nontrivial way on the transverse variable. For that purpose we construct a family of smooth solutions of the water waves equations which give arbitrarily small perturbations to a line solitary wave at the initial time and which after (long) times separate from the solitary wave (an its spatial translates) at some fixed distance, the distance being measured in some natural norm for the problem. More precisely, we prove an instability result in the L^2 norm which thus implies instability in the energy norm. Our result contains the fact that the solution remains smooth on a sufficiently long time scale where the instability can be observed.

The destabilization of one-dimensional stable patterns by transverse perturbations arises very often in dispersive equations. In the early 1970's, using the theory of integrable systems Zakharov [38] obtained the transverse instability of the soliton of the Korteweg -de Vries (KdV) equation considered as a one-dimensional solution of the (two-dimensional) Kadomtsev-Petviashvili-I (KP-I) equation. Since the KP-I equation can be obtained as a long-wave asymptotic model from the water waves system in the presence of enough surface tension (Bond number larger than 1/3), the situation considered by Zakharov can be thought as a strongly simplified model for the problem that we consider here namely the full water waves system with strong surface tension (Bond number larger than 1/3). Let us point out that the surface tension seems to have a destabilizing effect. Indeed, when the surface tension is weaker (Bond number smaller than 1/3), the asymptotic model in the same long-wave regime is the KP-II equation and for the KP-II equation, the linearized equation about the solitary wave has no unstable spectrum as shown in [5]. An interesting open question is therefore the study of the transverse stability of the line solitary waves constructed in [22] in the case of small surface tension (Bond number smaller than 1/3). Note that even for the model case of the KP-II equation the nonlinear stability is an interesting unsolved problem.

The main drawback of Zakharov approach is that a lot of dispersive equations like the water waves system that we want to study are not known to be completely integrable.

An important feature of most of the important models is that they are endowed with an Hamiltonian structure. This structure in the water-waves setting was exhibited by Zakharov [37]. Nevertheless, the general framework of Grillakis-Shatah-Strauss [17] which has been developed in order to prove stability or instability of constrained minimizers of the Hamiltonian, and has been successful for studying orbital stability of solitary waves in many dispersive models, does not seem to apply in transverse stability problems. The main reason is that the two-dimensional energy is infinite at the one-dimensional object.

In our previous works [31, 32], we developed an approach to study the transverse instability of solitary waves for Hamiltonian partial differential equations which applies in the situation considered by Zakharov and also to many other dispersive, not necessarily integrable, equations. This method inspired by a work by Grenier [16] in fluid mechanics consists in reducing the problem to the proof of linear instability for a family of one-dimensional problems by proving that linear instability implies nonlinear instability.

The water waves system with surface tension does not enter in the general framework of [32]. Among the main difficulties are the high level of nonlinearity in the equations which makes the study of the Cauchy problem for perturbations of the solitary wave non-trivial and the presence of a non-local term which does not allow to reduce the study of the equation linearized about the solitary wave to the study of ordinary differential equations. Nevertheless, the general philosophy of our approach can be used, more details on the description of our approach will be given in the end of this introduction.

The remaining part of the introduction is organized as follows. We first present the water waves system. Next, we describe the solitary waves as a special solution of the water waves system. Further, we give more details on the result of Mielke [26] about stability with respect to one dimensional perturbations. We then state our instability result with respect to two dimensional (transverse) perturbations. We end the introduction by explaining the general strategy behind our proof.

1.1. **The water waves system with surface tension.** We shall use the notation $Y = (X, z) \in \mathbb{R}^3$ with $X = (x, y) \in \mathbb{R}^2$. We consider the situation where the fluid domain which is unknown is defined by

$$\Omega_t = \{(X, z) \in \mathbb{R}^3 \,:\, -h < z < \eta(t, X)\},$$

where t is the time, h is a parameter defining the fixed bottom $z = -h$ and $z = \eta(t, X)$ is the equation of the free surface at time t. We denote by u the speed of the fluid. We consider the motion of an irrotational, incompressible fluid with constant density. This means that the velocity u of the fluid is given by $u = \nabla_Y \varphi = (\partial_x\phi, \partial_y\phi, \partial_z\phi)$ for some scalar function ϕ and hence we find that inside the fluid domain Ω_t,

$$\nabla_Y \cdot u = \Delta_Y \phi = (\partial_x^2 + \partial_y^2 + \partial_z^2)\phi(t, x, y, z) = 0, \quad \text{in} \quad \Omega_t. \tag{1.1}$$

On the boundaries of Ω_t, we make the usual assumption that no fluid particles cross the boundary. At the bottom of the fluid this reads

$$\partial_z\phi(t, x, y, -h) = 0 \tag{1.2}$$

and on the free surface, this yields the kinematic condition

$$\partial_t\eta(t, X) + \nabla_X\phi(t, X, \eta(t, X)) \cdot \nabla_X\eta(t, X) - \partial_z\phi(t, X, \eta(t, X)) = 0, \tag{1.3}$$

where we use the notation $\nabla_X \equiv (\partial_x, \partial_y)^t$. Finally, taking into account the surface tension to compute the pressure on the free surface, we find the Bernouilli law:

$$\partial_t\phi(t, X, \eta(t, X)) + \frac{1}{2}|\nabla_Y\phi(t, X, \eta(t, X))|^2 + g\eta(t, X) = b\nabla \cdot \frac{\nabla_X\eta(t, X)}{\sqrt{1 + |\nabla_X\eta(t, X)|^2}}\,, \tag{1.4}$$

where $\nabla_Y \equiv (\nabla_X, \partial_z)$. The coefficient b is the Bond number which measures the influence of the surface tension and g is the gravitational constant. The term $g\eta(t,X)$ is the trace of the gravitational force gz on the free surface.

It is classical to rewrite the system (1.1), (1.3), (1.4) as a system where all functions are evaluated on the free surface only. Let us next define the following Dirichlet-Neumann operator: for given $\eta(X)$ and $\varphi(X)$, we define $\phi(X,z)$ as the (well-defined) solution of the elliptic boundary value problem

$$(\partial_x^2 + \partial_y^2 + \partial_z^2)\phi = 0, \quad \text{in} \quad \{(X,z) : -h < z < \eta(X)\}, \quad \phi(X,\eta(X)) = \varphi(X), \quad \partial_z\phi(X,-h) = 0,$$

and we define the Dirichlet-Neumann operator as

$$G[\eta]\varphi \equiv \partial_z\phi(X,\eta(X)) - \nabla_X\eta(X)\cdot\nabla_X\phi(X,\eta(X)) = \sqrt{1+|\nabla_X\eta|^2}(\nabla_Y\phi(X,\eta(X))\cdot n(X)),$$

where

$$n(X) = \frac{1}{\sqrt{1+|\nabla_X\eta|^2}}(-\nabla_X\eta(X),1)$$

is the unit outward normal of the free surface at the point $z=\eta(X)$.

This allows to rewrite the system in terms of the functions evaluated on the free surface only. Set

$$\varphi(t,X) \equiv \phi(t,X,\eta(t,X)).$$

Then one directly checks that the water waves problem (1.1), (1.2), (1.3), (1.4) is reduced to the study of the following system

$$\partial_t\eta = G[\eta]\varphi, \tag{1.5}$$

$$\partial_t\varphi = -\frac{1}{2}|\nabla_X\varphi|^2 + \frac{1}{2}\frac{(G[\eta]\varphi + \nabla_X\varphi\cdot\nabla_X\eta)^2}{1+|\nabla_X\eta|^2} - g\eta + b\nabla_X\cdot\frac{\nabla_X\eta}{\sqrt{1+|\nabla_X\eta|^2}}. \tag{1.6}$$

As noticed by Zakharov [37], the system (1.5)-(1.6) has a canonical Hamiltonian structure

$$\partial_t\eta = \frac{\delta\mathcal{H}}{\delta\varphi}, \quad \partial_t\varphi = -\frac{\delta\mathcal{H}}{\delta\eta}$$

where the Hamiltonian $\mathcal{H}$ is the total energy given by

$$\mathcal{H}(\eta,\varphi) = \frac{1}{2}\int_{\mathbb{R}^2}\Big[G[\eta]\varphi\,\varphi + g\eta^2 + 2b(\sqrt{1+|\nabla\eta|^2}-1)\Big].$$

This is the sum of the kinetic energy, the gravitational potential energy and a surface energy due to stretching of the surface. The expression of the variational derivatives of $\mathcal{H}$ can be checked by easy calculation thanks to the following lemma (see [24, Theorem 3.20] for example).

Lemma 1.1. *For an integer $k\geq 2$, consider the map $\eta\mapsto G[\eta]\varphi$, acting between the Sobolev spaces $H^{k+1/2}(\mathbb{R}^2)$ and $H^{k-1/2}(\mathbb{R}^2)$. Then*

$$D_\eta G[\eta]\varphi\cdot\zeta = -G[\eta](\zeta Z) - \nabla\cdot\Big(\zeta(\nabla_X\varphi - Z\nabla_X\eta)\Big),$$

where Z, linear in φ and real valued, is defined by

$$Z = Z(\eta,\varphi) \equiv \frac{G[\eta]\varphi + \nabla_X\eta\cdot\nabla_X\varphi}{1+|\nabla_X\eta|^2}.$$

Note that because of the translational invariance in the problem, the momentum

$$\mathcal{P}(\eta,\varphi) = \int_{\mathbb{R}^2}\eta\varphi_x$$

is also a formally conserved quantity.

The Hamiltonian structure of (1.5)-(1.6) will be of crucial importance for many aspects of our analysis in particular for the choice of multipliers when performing energy estimates.

1.2. **The line solitary wave solution of the water waves system (1.5)-(1.6).** For $c \geq 0$, since we shall study solitary waves with speed c, we make a change of frame $X = (x, y, z) \mapsto (x - ct, y, z)$ which changes the dynamical equations (1.3), (1.4) into

$$\partial_t \eta(t, X) - c\partial_x \eta(t, X) + \nabla_X \phi(t, X, \eta(t, X)) \cdot \nabla_X \eta(t, X) - \partial_z \phi(t, X, \eta(t, X)) = 0$$

and

$$\partial_t \phi(t, X, \eta(t, X)) - c\partial_x \phi(t, X, \eta(t, X)) + \frac{1}{2}|\nabla_Y \phi(t, X, \eta(t, X))|^2 + g\eta(t, X) = b\nabla_X \cdot \frac{\nabla_X \eta(t, X)}{\sqrt{1 + |\nabla_X \eta(t, X)|^2}}.$$

By using again the Dirichlet-Neumann operator, the equations (1.5), (1.6) become

$$\partial_t \eta = c\,\partial_x \eta + G[\eta]\varphi, \tag{1.7}$$

$$\partial_t \varphi = c\,\partial_x \varphi - \frac{1}{2}|\nabla_X \varphi|^2 + \frac{1}{2}\frac{(G[\eta]\varphi + \nabla_X \varphi \cdot \nabla_X \eta)^2}{1 + |\nabla_X \eta|^2} - g\eta + b\nabla_X \cdot \frac{\nabla_X \eta}{\sqrt{1 + |\nabla_X \eta|^2}} \tag{1.8}$$

where φ is again defined as $\varphi(t, X) = \phi(t, X, \eta(t, X))$. A solitary wave with speed c becomes a stationary solution (i.e. independent of t) of (1.7), (1.8). To study the existence of such solitary waves, it is classical to introduce a non-dimensional version of the equations. Let us perform the change of variable

$$\eta(t, X) = h\,\tilde{\eta}\Big(\frac{c}{h}t, \frac{1}{h}X\Big), \quad \phi(t, X, z) = c\,h\,\tilde{\phi}\Big(\frac{c}{h}t, \frac{1}{h}X, \frac{1}{h}z\Big).$$

Then the equations satisfied by $\tilde{\eta}$, $\tilde{\phi}$ which for the sake of simplicity will still be denoted by η, ϕ are

$$\partial_t \eta(t, X) - \partial_x \eta(t, X) + \nabla_X \phi(t, X, \eta(t, X)) \cdot \nabla_X \eta(t, X) - \partial_z \phi(t, X, \eta(t, X)) = 0$$

and

$$\partial_t \phi(t, X, \eta(t, X)) - \partial_x \phi(t, X, \eta(t, X)) + \frac{1}{2}|\nabla_Y \phi(t, X, \eta(t, X))|^2 + \alpha\eta(t, X) = \beta\nabla_X \cdot \frac{\nabla_X \eta(t, X)}{\sqrt{1 + |\nabla_X \eta(t, X)|^2}},$$

where the fluid domain is now $\{(X, z) : -1 < z < \eta(t, X)\}$ and

$$\alpha = \frac{gh}{c^2}, \quad \beta = \frac{b}{hc^2}.$$

Note that the elliptic equation (1.1) is not changed. The equations formulated on the free surface thus become

$$\partial_t \eta = \partial_x \eta + G[\eta]\varphi, \tag{1.9}$$

$$\partial_t \varphi = \partial_x \varphi - \frac{1}{2}|\nabla_X \varphi|^2 + \frac{1}{2}\frac{(G[\eta]\varphi + \nabla_X \varphi \cdot \nabla_X \eta)^2}{1 + |\nabla_X \eta|^2} - \alpha\eta + \beta\nabla_X \cdot \frac{\nabla_X \eta}{\sqrt{1 + |\nabla_X \eta|^2}}. \tag{1.10}$$

The Hamiltonian is now given by

$$H(\eta, \varphi) = \frac{1}{2}\int_{\mathbb{R}^2} \Big[G[\eta]\varphi\,\varphi + \alpha\eta^2 + 2\beta(\sqrt{1 + |\nabla\eta|^2} - 1) - 2\,\eta\,\partial_x \varphi\Big].$$

In terms of the parameters α and β, we have the following existence result (see [6]) concerning stationary solutions of (1.9)-(1.10) (or equivalently solitary wave solutions of the original problem (1.5)-(1.6)).

Theorem 1.2 (Amick-Kirchgässner [6]). *Suppose that $\alpha = 1 + \varepsilon^2$ and $\beta > 1/3$. Then there exists ε_0 such that for every $\varepsilon \in (0, \varepsilon_0)$ there is a stationary solution $(\eta_\varepsilon(x), \varphi_\varepsilon(x))$ (i.e. also independent of y) of (1.9)-(1.10) under the form*

$$\eta_\varepsilon(x) = \varepsilon^2 \Theta(\varepsilon x, \varepsilon), \quad \varphi_\varepsilon(x) = \varepsilon \Phi(\varepsilon x, \varepsilon),$$

where Θ and Φ satisfy:

$$\exists\, d > 0, \quad \forall\, \alpha \in \mathbb{N}, \quad \exists\, C_\alpha > 0, \quad \forall\, (x, \varepsilon) \in \mathbb{R} \times (0, \varepsilon_0), \quad |(\partial_x^\alpha \Theta)(x, \varepsilon)| \le C_\alpha e^{-d|x|}$$

and

$$\exists\, d > 0, \quad \forall\, \alpha \ge 1, \quad \exists\, C_\alpha > 0, \quad \forall\, (x, \varepsilon) \in \mathbb{R} \times (0, \varepsilon_0), \quad |(\partial_x^\alpha \Phi)(x, \varepsilon)| \le C_\alpha e^{-d|x|}.$$

Observe that the speeds of the solitary waves of (1.5)-(1.6) built in the above result are close to $\sqrt{gh}$ which is independent of ε.

The profiles $\Theta(\xi, \varepsilon)$ and $\Phi(\xi, \varepsilon)$ have smooth expansions in ε. In particular for $\varepsilon = 0$, we find

$$\Theta(\xi, 0) = -\cosh^{-2}\Big(\frac{\xi}{2(\beta - 1/3)^{1/2}}\Big) \tag{1.11}$$

and hence we recover the KdV solitary wave.

1.3. **Stability with respect to one-dimensional perturbations.** A very natural question is to study the stability of the solitary wave solutions obtained in Theorem 1.2. Because of the invariance of the problem with respect to spatial translations, usual Lyapounov stability cannot hold and thus it is natural to study the stability of the solitary wave modulo these translations (orbital stability). It turns out that under one-dimensional perturbations the solitary waves of Amick-Kirchgässner are (orbitally) stable.

Let us fix the functional setting. We define the space $Z(\mathbb{R})$ as $Z(\mathbb{R}) = H^1(\mathbb{R}) \times H_*^{\frac{1}{2}}(\mathbb{R})$ where

$$H_*^{\frac{1}{2}}(\mathbb{R}) = \Big\{\varphi, \quad \|\varphi\|^2_{H_*^{\frac{1}{2}}} = \int_{\mathbb{R}} |\xi| \tanh|\xi|\, |\hat{\varphi}(\xi)|^2 < +\infty, \Big\}_{/\mathbb{R}}$$

which means that we do not distinguish functions that just differ by a constant. Note that the control given by the $\|\cdot\|_{H_*^{\frac{1}{2}}}$ semi-norm in the low frequencies is worse than the one given by the usual homogeneous $\dot{H}^{\frac{1}{2}}$ semi-norm. This is the natural semi-norm associated to the Dirichlet-Neumann operator and thus $Z(\mathbb{R})$ is the natural space associated to the Hamiltonian. Nevertheless, to make this statement rigorous, we need a little bit more control on the regularity of the surface. We thus also introduce for $R > 1$ the subspace

$$Z_R(\mathbb{R}) = \Big\{U = (\eta, \varphi) \in Z(\mathbb{R}), \quad -1 + \frac{1}{R} \le \eta(x) \le R, \quad \|\eta_x\|_{L^\infty} \le R\Big\}.$$

The result of [26] reads as follows.

Theorem 1.3 (Mielke [26], 1d stability). *Let α, β and ε_0 be as in Theorem 1.2. Then there exists $\varepsilon_1 > 0$ such that for every $\varepsilon \in (0, \varepsilon_1]$ and $R > 1$, the solitary wave $(\eta_\varepsilon, \varphi_\varepsilon)$ is conditionally stable in the following sense.*

For every $\kappa > 0$, there exists $\delta > 0$ such that : if $U = (\eta, \varphi) : [0, T) \to Z_R(\mathbb{R})$ is a continuous solution of (1.5), (1.6)*, which preserves the Hamiltonian $\mathcal{H}$ and the momentum $\mathcal{P}$ and satisfies $\|U(0) - (\eta_\varepsilon, \varphi_\varepsilon)\|_Z \le \delta$ then it satisfies*

$$\inf_{x_0 \in \mathbb{R}} \|U(t, \cdot - x_0) - (\eta_\varepsilon, \varphi_\varepsilon)\|_Z \le \kappa, \quad \forall t \in [0, T).$$

As stated by Mielke, the assumption that ε is sufficiently small can be replaced by assuming that a family of solitary waves depending smoothly on the speed exists and that the list of spectral assumptions and the condition on the moment of instability necessary in the framework of [17] hold.

1.4. **Main result: transverse nonlinear instability.** The situation drastically changes if one considers $2d$ (transverse) perturbations. The main result of this paper is that the solitary wave solutions obtained in Theorem 1.2 are (orbitaly) unstable when submitted to two-dimensional localized perturbations (transverse instability). Here is the precise statement.

Theorem 1.4 (Transverse instability)**.** *Let α, β and ε as in Theorem 1.2. There exists $\varepsilon_1 > 0$ such that for every $\varepsilon \in (0, \varepsilon_1]$ the following holds true.*

For every $s \geq 0$, there exists $\kappa > 0$ such that for every $\delta > 0$, we can find an initial data $(\eta_0^\delta(x,y), \varphi_0^\delta(x,y))$ and a time $T^\delta \sim |\log \delta|$ such that

$$\|(\eta_0^\delta(x,y), \varphi_0^\delta(x,y)) - (\eta_\varepsilon(x), \varphi_\varepsilon(x))\|_{H^s(\mathbb{R}^2)\times H^s(\mathbb{R}^2)} \leq \delta$$

and there exists a solution $(\eta^\delta(t,x,y), \varphi^\delta(t,x,y))$ of the water waves equation (1.9)-(1.10) with data $(\eta_0^\delta, \varphi_0^\delta)$, defined on $[0, T^\delta]$ and satisfying

$$\inf_{a\in\mathbb{R}} \|(\eta^\delta(T^\delta, x, y), \varphi^\delta(T^\delta, x, y)) - (\eta_\varepsilon(x-a), \varphi_\varepsilon(x-a))\|_{L^2(\mathbb{R}^2)\times L^2(\mathbb{R}^2)} > \kappa.$$

Let us give a few comments on this result.

The instability is stated in the L^2 norm. This thus implies an instability in the energy norm $H^1 \times H^{\frac{1}{2}}$. We shall actually establish the stronger result that the L^2 distance of the solution to all functions depending on x only is at time T^δ larger than κ.

As in Theorem 1.3, the assumption that ε is sufficiently small can be replaced by the same assumptions as in [26]. Namely, we need that the solitary wave exists and that the linearization of the one-dimensional equation about the solitary wave verifies some spectral assumptions. Note that we only need the existence of the solitary wave and some stability properties of the one-dimensional problem without any additional assumption in order to get the transverse instability i.e ε_1 is the same in Theorems 1.3 and 1.4.

Let us remark that our theorem is not conditional: we establish the existence of the solution on $[0, T^\delta]$ which is already a non trivial part of the statement.

1.5. **Outlines of the paper.** To prove Theorem 1.4, we shall construct the solution $U^\delta = (\eta^\delta, \varphi^\delta)$ of (1.9), (1.10) under the form

$$U^\delta = U_\varepsilon + U^a + V, \quad U_\varepsilon = (\eta_\varepsilon, \varphi_\varepsilon)^t$$

where following the approach of [16], U^a is an exponentially growing solution driven by the linear instability and V is a corrector that we add in order to get an exact solution of the nonlinear equation. There are three main parts in the paper. In the first part, we study the linearized water waves equations about the solitary wave, the aim is to construct the leading part of U^a as an exponentially growing solution of the linearized equation with the maximal growth rate. The second step is the construction of the remaining part of U^a where we describe how the linear instability interacts with the nonlinear term. The last step is the construction of the correction term V where we need to study a nonlinear problem. Here are more details:

- As a preliminary step, in Section 2, we study the structure of the water waves equations linearized about the solitary waves $(\eta_\varepsilon, \varphi_\varepsilon)^t$. By using the fact that the solitary waves do not depend on the transverse variable, we can Fourier transform the linearized equation in the transverse variable to reduce the problem to the study of a family of linear equations indexed by the transverse frequency parameter $k \in \mathbb{R}$:

$$\partial_t U = JL(k)U \tag{1.12}$$

where $L(k)$ is a symmetric operator and $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. In the expression of the operator $L(k)$ arises the "Fourier transform" of the Dirichlet-Neumann operator $G_{\varepsilon,k}$ defined as

$$G[\eta_\varepsilon]\big(f(x)e^{iky}\big) = e^{iky} G_{\varepsilon,k}\big(f(x)\big).$$

In order to understand the main properties of the linearized equation (1.12), we first need to study carefully $G_{\varepsilon,k}$. This is the aim of Section 3. The estimates that we establish are rather classical when k is fixed, we refer for example to [24], [3], the main novelty is that we need to track carefully the dependence in k (especially when k is close to zero) in the estimates. We also point out the elementary but very useful property that $G_{\varepsilon,k}$ as a symmetric operator depends on $|k|$ in a monotonous way.

- In Section 4, we study the properties of $L(k)$. We establish that it has a self-adjoint realization on $L^2 \times L^2$ with domain $H^2 \times H^1$ and study its spectrum. We first get (Proposition 4.2) that its essential spectrum is contained in $[c_k, +\infty)$ where $c_k \geq 0$ and $c_k > 0$ if $k \neq 0$. Next, in Proposition 4.8, we prove that for ε sufficiently small, $L(k)$ has at most one negative eigenvalue for every k. Note that in the case $k = 0$ the spectrum of $L(k)$ can be described by using the spectrum of the KdV equation linearized about the KdV solitary wave as shown by Mielke in [26].
- In Section 5, we study the operator $JL(k)$. We prove (Proposition 5.5) that its essential spectrum is included in $i\mathbb{R}$ and locate its possible unstable (i.e. with positive real parts) eigenvalues in Proposition 5.2. Finally, in Theorem 5.3, we prove the linear instability: we show that for some $k \neq 0$, $JL(k)$ has an unstable eigenvalue. This last result is known, it was obtained in [28], [18], [7] for example by using different formulations of the water waves equation. The proof that we get here is very simple, it just relies on the monotonous dependence of $L(k)$ in k and a bifurcation argument based on the Lyapounov-Schmidt method. An important consequence of this part is that we get the existence of a most unstable eigenmode i.e an eigenvalue $\sigma(k_0)$ of $JL(k_0)$ such that

$$\text{Re } \sigma(k_0) = \sup \big\{ \text{Re } \sigma, \quad \exists k,\, \sigma \in \sigma\big(JL(k)\big) \big\}.$$

- Once these main properties are established, we are able to construct the unstable approximate solution $U^a = (\eta^a, \varphi^a)$. From the spectral properties of $JL(k)$, we take the first part U^0 of U^a (see Proposition 6.1) under the form

$$U^0 = \int_I e^{\sigma(k)t} e^{iky} U(k)\, dk$$

where $\sigma(k)$ is an analytic curve passing through $\sigma(k_0)$.
- The next step is to construct U^a (Proposition 6.3). We look for U^a under the form

$$U^a = \delta \sum_{j=0}^{M} \delta^j U^j$$

where each term U^j must be bounded from above by $\sim e^{\sigma_0 (j+1)t}$ with $\sigma_0 = \text{Re } \sigma(k_0)$. They are solutions of linear equations with source terms. The crucial property that is thus needed is an accurate H^s estimate for the semi-group of $JL(k)$. Since $JL(k)$ is not sectorial some work is needed to establish it. Here, we use the Laplace transform. To control the high time frequencies, we use energy estimates based on the Hamiltonian structure of the equation and the properties of $L(k)$. For the bounded frequencies, we use abstract arguments based on the knowledge of the spectrum of $JL(k)$.
- The last step is to construct the correction term V which solves a nonlinear water-waves equation. This is the aim of Section 7.

The local well-posedness for the water waves equation has been much studied recently, we refer for example to [36], [24], [27], [9], [25], [11], [34]. Here, we want to prove that there exists a smooth solution of the water waves equation in the vicinity of the approximate unstable solution which remains smooth on a sufficiently long interval of time. Moreover, we want a precise estimate between the exact and the approximate solution in order to get the instability result. For this reason the approaches like [24] or [27] which are based on the Nash-Moser's scheme are not suitable for our purpose. It was noticed in [21] that when there is no surface tension, the water waves system has a quasilinear structure once we have applied three space derivatives on it. When there is surface tension, the main difficulty is that the commutator between a space derivative and the term coming from the surface tension contains too many derivatives to be considered as a remainder. This situation arises classically in the study of high order wave equations for example

$$\partial_{tt}u = -|D|^{\frac{3}{2}}\big(a(u)|D|^{\frac{3}{2}}u\big), \quad a \geq a_0.$$

Note that in 1-D the water wave problem in Lagrangian coordinates is indeed very close to this situation (see [33] for example). For such high order wave equations, a good candidate in order to get H^s type estimates is to apply powers of the operator $|D|^{\frac{3}{2}}\big(a(u)|D|^{\frac{3}{2}}u\big)$ to the equation. This is the approach chosen in the study of the water waves system in [34].

Here, to handle this difficulty we shall use a slightly different approach which is based on the use of time derivatives: the energies that we use involve simultaneous space and time derivatives of the unknown. The basic block in the construction of our energies comes from the Hamiltonian structure of the system, nevertheless, we also need to add some lower order terms in order to cancel some commutators. This approach yields slightly simpler commutators to compute and allows to get a quasilinear form of the system when there is surface tension. Note that our argument provides the well-posedness (without Nash-Moser's scheme) of the water waves with surface tension (a result already obtained in [27] via Nash-Moser's scheme). A technical difficulty in this section is that we need H^s estimates of terms like

$$\Big(G[\eta_\varepsilon + \eta^a + \eta] - G[\eta_\varepsilon + \eta^a]\Big) \cdot (\varphi_\varepsilon + \varphi^a).$$

This yields because of the solitary wave (since η_ε, φ_ε and their derivatives are not in $H^s(\mathbb{R}^2)$) that we need to study the Dirichlet Neumann operator in a non H^s framework. The final argument to get the instability is the one of [16].

2. The linearized water waves equation about the solitary wave $(\eta_\varepsilon, \varphi_\varepsilon)$

In this section, we shall study the structure of the linearized water waves equations about the solitary wave.

In view of Lemma 1.1, in order to express the linear equation arising from the linearization of (1.9), (1.10) about the solitary wave $Q_\varepsilon = (\eta_\varepsilon, \varphi_\varepsilon)$, it is convenient to use the notation

$$Z_\varepsilon \equiv Z[\eta_\varepsilon, \varphi_\varepsilon], \quad \nabla_X\varphi_\varepsilon - Z_\varepsilon\nabla_X\eta_\varepsilon \equiv \begin{pmatrix} v_\varepsilon \\ 0 \end{pmatrix}.$$

Thus

$$v_\varepsilon = \partial_x\varphi_\varepsilon - \frac{G[\eta_\varepsilon]\varphi_\varepsilon + \partial_x\eta_\varepsilon\partial_x\varphi_\varepsilon}{1+|\partial_x\eta_\varepsilon|^2}\partial_x\eta_\varepsilon\,.$$

We also introduce the operator (of Laplace-Beltrami type) P_ε defined by

$$P_\varepsilon\eta \equiv \beta\nabla_X \cdot \Big[\frac{\nabla_X\eta}{(1+|\partial_x\eta_\varepsilon|^2)^{\frac{1}{2}}} - \frac{(\nabla_X\eta_\varepsilon\cdot\nabla_X\eta)\nabla_X\eta_\varepsilon}{(1+|\partial_x\eta_\varepsilon|^2)^{\frac{3}{2}}}\Big].$$

Since the solitary wave is one-dimensional, we observe that

$$(\nabla_X \eta_\varepsilon \cdot \nabla_X \eta) \nabla_X \eta_\varepsilon = \begin{pmatrix} (\partial_x \eta_\varepsilon)^2 \partial_x \eta \\ 0 \end{pmatrix},$$

therefore, the linearization of (1.9)-(1.10) about $(\eta_\varepsilon, \varphi_\varepsilon)$ reads

$$\begin{aligned} \partial_t \eta &= \partial_x \eta + G[\eta_\varepsilon]\varphi - G[\eta_\varepsilon](Z_\varepsilon \eta) - \partial_x(v_\varepsilon \eta), \\ \partial_t \varphi &= \partial_x \varphi + P_\varepsilon \eta - v_\varepsilon \partial_x \varphi + Z_\varepsilon G[\eta_\varepsilon]\varphi - Z_\varepsilon G[\eta_\varepsilon](Z_\varepsilon \eta) - (\alpha + Z_\varepsilon \partial_x v_\varepsilon)\eta \,. \end{aligned}$$

This linear equation has a canonical Hamiltonian structure and can be written as

$$\partial_t \begin{pmatrix} \eta \\ \varphi \end{pmatrix} = J\Lambda \begin{pmatrix} \eta \\ \varphi \end{pmatrix}, \tag{2.1}$$

where

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is skew-symmetric and

$$\Lambda = \begin{pmatrix} -P_\varepsilon + \alpha + Z_\varepsilon G[\eta_\varepsilon]\big(Z_\varepsilon \,\cdot\,\big) + Z_\varepsilon \partial_x v_\varepsilon & (v_\varepsilon - 1)\partial_x - Z_\varepsilon G[\eta_\varepsilon] \\ -\partial_x((v_\varepsilon - 1)\cdot) - G[\eta_\varepsilon]\big(Z_\varepsilon \,\cdot\,\big) & G[\eta_\varepsilon] \end{pmatrix}$$

is a symmetric operator. As noticed by Lannes in [24], we get a more tractable expression of the linearized equation if we introduce the change of unknowns

$$V_1 = \eta, \quad V_2 = \varphi - Z_\varepsilon \eta. \tag{2.2}$$

Indeed, if (η, φ) solves the system (2.1), then (V_1, V_2) solves the system

$$\begin{aligned} \partial_t V_1 &= G[\eta_\varepsilon]V_2 - \partial_x((v_\varepsilon - 1)V_1), \\ \partial_t V_2 &= P_\varepsilon V_1 - (v_\varepsilon - 1)\partial_x V_2 - (\alpha + (v_\varepsilon - 1)\partial_x Z_\varepsilon)V_1. \end{aligned}$$

As noticed in [1], this change of unknown is linked with the "good unknown" of Alinhac [2]. The last system can be written in the canonical Hamiltonian form

$$\partial_t \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = JL \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, \tag{2.3}$$

where the symmetric operator L is defined as follows

$$L = \begin{pmatrix} -P_\varepsilon + \alpha + (v_\varepsilon - 1)\partial_x Z_\varepsilon & (v_\varepsilon - 1)\partial_x \\ -\partial_x((v_\varepsilon - 1)\cdot) & G[\eta_\varepsilon] \end{pmatrix}.$$

Since η_ε does not depend on y, the study of (2.3) can be simplified by using the Fourier transform in y. Indeed, if for some $k \in \mathbb{R}$,

$$V_1(x,y) = e^{iky}W_1(x), \quad V_2(x,y) = e^{iky}W_2(x) \tag{2.4}$$

then

$$L \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = e^{iky} L(k) \begin{pmatrix} W_1 \\ W_2 \end{pmatrix},$$

where the symmetric operator $L(k)$ is defined as

$$L(k) = \begin{pmatrix} -P_{\varepsilon,k} + \alpha + (v_\varepsilon - 1)\partial_x Z_\varepsilon & (v_\varepsilon - 1)\partial_x \\ -\partial_x((v_\varepsilon - 1)\cdot) & G_{\varepsilon,k} \end{pmatrix}$$

with

$$P_{\varepsilon,k} u = \beta\Big(\partial_x\big((1 + (\partial_x \eta_\varepsilon)^2)^{-\frac{3}{2}} \partial_x u\big) - k^2 (1 + (\partial_x \eta_\varepsilon)^2)^{-\frac{1}{2}} u\Big)$$

and $G_{\varepsilon,k}$ is such that

$$G[\eta_\varepsilon](f(x)\exp(iky)) = \exp(iky) G_{\varepsilon,k}(f(x)) \,. \tag{2.5}$$

The fact that $G_{\varepsilon,k} = G_{\varepsilon,k}(x, D_x, k)$ is independent of y follows directly from the definition of the Dirichlet-Neumann operator. Note that $-P_{\varepsilon,k} + \alpha$ is a positive operator: there exists $c > 0$ independent of $k \in \mathbb{R}$ such that for every $u \in H^1(\mathbb{R})$,

$$\int_{\mathbb{R}} ((-P_{\varepsilon,k}u + \alpha u)\bar{u} \geq c\big(|u|^2_{H^1(\mathbb{R})} + (k^2+1)|u|^2_{L^2(\mathbb{R})}\big). \tag{2.6}$$

Note that, for $k \in \mathbb{R}$, we can also define the operator $\Lambda(k)$ associated to Λ acting on functions depending on x only as

$$\Lambda\Big(e^{iky}\begin{pmatrix} V_1(x) \\ V_2(x) \end{pmatrix}\Big) = e^{iky}\Lambda(k)\begin{pmatrix} V_1(x) \\ V_2(x) \end{pmatrix}.$$

We find for $\Lambda(k)$ the expression

$$\Lambda(k) = \begin{pmatrix} -P_{\varepsilon,k} + \alpha + Z_\varepsilon G_{\varepsilon,k}\big(Z_\varepsilon\,\cdot\big) + Z_\varepsilon \partial_x v_\varepsilon & (v_\varepsilon - 1)\partial_x - Z_\varepsilon G_{\varepsilon,k} \\ -\partial_x((v_\varepsilon - 1)\cdot) - G_{\varepsilon,k}\big(Z_\varepsilon\,\cdot\big) & G_{\varepsilon,k} \end{pmatrix}.$$

Due to the change of unknown (2.2), we have the relation

$$JL(k) = P^{-1}J\Lambda(k)P$$

where

$$P = \begin{pmatrix} 1 & 0 \\ Z_\varepsilon & 1 \end{pmatrix}, \quad Q = P^{-1} = \begin{pmatrix} 1 & 0 \\ -Z_\varepsilon & 1 \end{pmatrix}.$$

Since P and P^{-1} are just smooth matrices, $JL(k)$ and $J\Lambda(k)$ have thus the same spectrum. Moreover, we also have that $L(k)$ and $\Lambda(k)$ are linked through

$$L(k) = P^\star \Lambda(k) P, \quad \Lambda(k) = Q^\star L(k) Q \tag{2.7}$$

therefore, it is also possible to relate spectral properties of $L(k)$ and $\Lambda(k)$ via the analysis of the corresponding quadratic forms.

In the next section, we shall establish some useful properties on the Dirichlet-Neumann operator $G_{\varepsilon,k}$ and on the spectrum of $L(k)$.

3. Study of the Dirichlet to Neumann operator $G_{\varepsilon,k}$

In this section, we shall study the basic properties of $G_{\varepsilon,k}$. An elementary but very useful property that we establish is the monotonicity property of $G_{\varepsilon,k}$ with respect to k. The proofs of most of the other properties are inspired by similar considerations in [3, 24], the point here being to track the dependence with respect to k in the estimates.

Note that, because of the definition (2.5), we need to work with complex valued functions. For complex valued functions, we shall denote the complex L^2 scalar product as

$$(u, v) = \int u(x)\overline{v(x)}\,dx. \tag{3.1}$$

We shall use slightly abusively the same notation for the scalar product of $L^2 \times L^2$, thus for $U = (U_1, U_2)$, $V = (V_1, V_2)$ in $L^2 \times L^2$, we define

$$(U, V) = (U_1, V_1) + (U_2, V_2).$$

Note that we have

$$\mathrm{Re}\,(JU, U) = 0, \quad \forall U \in L^2 \times L^2. \tag{3.2}$$

We shall first prove the following statement.

Proposition 3.1. *For every $\varepsilon > 0$, we have the following properties:*

i) $G_{\varepsilon,k}$ *is symmetric :*

$$\big(G_{\varepsilon,k}u, v\big) = \big(u, G_{\varepsilon,k}v\big), \quad \forall\, u,\, v \in H^{\frac{1}{2}}(\mathbb{R})\,.$$

ii) *If* $|k_1| > |k_2|$, *then* $G_{\varepsilon,k_1} - G_{\varepsilon,k_2}$ *is a positive definite operator :*

$$\big(G_{\varepsilon,k_1}u, u\big) > \big(G_{\varepsilon,k_2}u, u\big), \quad \forall\, u \in H^{\frac{1}{2}}(\mathbb{R}),\, u \neq 0.$$

iii) *There exist* $c > 0$ *and* $C > 0$ *such that for every* $k \in \mathbb{R}$, *we have*

$$\big|\big(G_{\varepsilon,k}u, v\big)\big| \leq C\Big|\frac{\sqrt{D_x^2+k^2}}{\big(1+\sqrt{D_x^2+k^2}\big)^{\frac{1}{2}}}u\Big|_{L^2}\Big|\frac{\sqrt{D_x^2+k^2}}{\big(1+\sqrt{D_x^2+k^2}\big)^{\frac{1}{2}}}v\Big|_{L^2}, \quad \forall\, u,\, v \in H^{\frac{1}{2}}(\mathbb{R}), \tag{3.3}$$

$$\big(G_{\varepsilon,k}u, u\big) \geq c\Big|\frac{\sqrt{D_x^2+k^2}}{\big(1+\sqrt{D_x^2+k^2}\big)^{\frac{1}{2}}}u\Big|^2_{L^2}, \quad \forall\, u \in H^{\frac{1}{2}}(\mathbb{R})\,. \tag{3.4}$$

Note that the estimates of the above proposition have a sharp dependence in k. In particular, (3.3), (3.4) are uniform in k, $k \in \mathbb{R}$. We do not care on the dependence of these estimates in ε.

Proof of Proposition 3.1. We first prove *i*). We recall that by definition, we have

$$G_{\varepsilon,k}(u)(x) = \sqrt{1+(\partial_x\eta_\varepsilon(x))^2}(\nabla_{x,z}\phi_k^u(x,\eta_\varepsilon(x))\cdot n(x)),$$

where $\phi_k^u(x,z)$ is the solution of the elliptic problem

$$(\partial_x^2 - k^2 + \partial_z^2)f = 0, \quad -1 < z < \eta_\varepsilon(x),\ x \in \mathbb{R} \quad \partial_z f(x,-1) = 0 \tag{3.5}$$

such that

$$f(x,\eta_\varepsilon(x)) = u(x), \quad x \in \mathbb{R}\,. \tag{3.6}$$

The identity i) will be a simple consequence of the Green formula. Indeed, let us set $D = \{(x,z) : -1 < z < \eta_\varepsilon(x)\}$ and $\Sigma = \{z = \eta_\varepsilon(x)\} \cup \{z = -1\}$ and consider ϕ_k^u, ϕ_k^v the solutions of (3.5), (3.6) with respective traces u and v on the upper boundary. Then, by definition, we have

$$(G_{\varepsilon,k}u, v) = \int_{\mathbb{R}} G_{\varepsilon,k}u(x)\,\overline{v(x)}\,dx = \int_\Sigma \frac{\partial\phi_k^u}{\partial n}(\tau)\overline{\phi_k^v(\tau)}d\Sigma(\tau),$$

where $d\Sigma(\tau)$ is the volume element of the surface $z = \eta_\varepsilon(x)$. Consequently, since $\partial_z\phi_k^u(x,-1) = \partial_z\phi_k^v(x,-1) = 0$, the Green formula and the equations satisfied by ϕ_k^u, ϕ_k^v yield

$$(G_{\varepsilon,k}u, v) = \int_D \Big(\nabla_{x,z}\phi_k^u \cdot \overline{\nabla_{x,z}\phi_k^v} + k^2\phi_k^u\overline{\phi_k^v}\Big)dxdz = (u, G_{\varepsilon,k}v). \tag{3.7}$$

This proves i).

Let us now prove ii). We first observe that if u is real then $G_{\varepsilon,k}u$ is also real. Therefore, if $u = u_1 + iu_2$ with real valued u_1 and u_2, we have that

$$(G_{\varepsilon,k_1}u, u) = (G_{\varepsilon,k_1}u_1, u_1) + (G_{\varepsilon,k_1}u_2, u_2).$$

Consequently, we can assume that u is real valued for the proof. Thanks to (3.7), we have

$$\begin{aligned}
(G_{\varepsilon,k_1}u, u) &= \int_D \Big|\nabla_{x,z}\phi_{k_1}^u|^2 + k_1^2|\phi_{k_1}^u|^2\Big)dxdz \\
&= \int_D \Big(|\nabla_{x,z}\phi_{k_1}^u|^2 + k_2^2|\phi_{k_1}^u|^2\Big)dxdz + (k_1^2 - k_2^2)\int_D |\phi_{k_1}^u|^2\,dxdz \\
&> \int_D \Big(|\nabla_{x,z}\phi_{k_1}^u|^2 + k_2^2|\phi_{k_1}^u|^2\Big)dxdz
\end{aligned}$$

since $|k_1| > |k_2|$ and $u \neq 0$. Next, since $\phi^u_{k_1}$ and $\phi^u_{k_2}$ verify the same boundary conditions, we have thanks to the variational characterization of $\phi^u_{k_2}$ that

$$\int_D |\nabla_{x,z}\phi^u_{k_1}|^2 + k_2^2 \int_D |\phi^u_{k_1}|^2 \geq \int_D |\nabla_{x,z}\phi^u_{k_2}|^2 + k_2^2 \int_D |\phi^u_{k_2}|^2 .$$

Consequently, by using again (3.7), we get

$$(G_{\varepsilon,k_1}u,u) > \int_D |\nabla_{x,z}\phi^u_{k_2}|^2 + k_2^2 \int_D |\phi^u_{k_2}|^2 = (G_{\varepsilon,k_2}u,u).$$

This proves ii).

We can now prove iii). Note that here, since η_ε is smooth and fixed, we do not care on the way the estimates depend on the regularity of η_ε.

Next, to prove (3.4), (3.3), it is convenient to rewrite the elliptic problem (3.5) in a flat domain. We can define implicitly a function ψ^u_k on the flat domain $\mathcal{S} = \mathbb{R} \times (-1,0)$ by

$$\phi^u_k(x,z) = \psi^u_k\Big(x, \frac{z-\eta_\varepsilon(x)}{1+\eta_\varepsilon(x)}\Big), \quad x \in \mathbb{R}, \quad -1 < z < \eta_\varepsilon(x).$$

Since we have by the chain rule

$$\nabla \phi^u_k(x,z) = M(x,z)\nabla\psi^u_k(x, m(x,z)), \tag{3.8}$$

where

$$m(x,z) = \frac{z-\eta_\varepsilon(x)}{1+\eta_\varepsilon(x)}, \quad M(x,z) = \begin{pmatrix} 1 & \partial_x m \\ 0 & \partial_z m \end{pmatrix},$$

we also get by using that the divergence is the L^2 adjoint of the gradient that for a vector field $u(x,z)$ on D such that

$$u(x,z) = v(\Phi(x,z)), \quad \Phi(x,z) = (x, m(x,z))$$

we have

$$\nabla \cdot u(x,z) = \det\left(D\Phi(x,z)\right) \nabla_Y \cdot \Big(\det(D\Phi^{-1}(Y)) M\big(\Phi^{-1}(Y)\big)^* v(Y)\Big)_{/Y=\Phi(x,z)}.$$

This allows to get that

$$\Delta\phi^u_k = \nabla\cdot\nabla\phi^u_k = \Delta_g \psi^u_k$$

where the operator Δ_g defined as

$$\Delta_g(\psi) = (\det(g))^{-1/2}\mathrm{div}\Big((\det(g))^{1/2} g^{-1}\nabla\psi\Big) \tag{3.9}$$

is the Laplace Beltrami operator associated to the metric g which is defined through its inverse g^{-1} by

$$g^{-1}(x,z) \equiv \begin{pmatrix} 1 & -\frac{\partial_x\eta_\varepsilon(x)(z+1)}{1+\eta_\varepsilon(x)} \\ -\frac{\partial_x\eta_\varepsilon(x)(z+1)}{1+\eta_\varepsilon(x)} & \frac{1+(z+1)^2(\partial_x\eta_\varepsilon(x))^2}{(1+\eta_\varepsilon(x))^2} \end{pmatrix} = M\big(\Phi^{-1}(x,z)\big)^* M\big(\Phi^{-1}(x,z)\big), \quad (x,z) \in \mathcal{S}.$$

Consequently, if ϕ^u_k solves

$$(\partial_x^2 - k^2 + \partial_z^2)\phi = 0, \quad x \in \mathbb{R}, \quad -1 < z < \eta_\varepsilon(x),$$

with boundary conditions $\phi(x,\eta_\varepsilon(x)) = u(x)$, $\partial_z\phi(x,-1) = 0$ then ψ^u_k, solves

$$(-\Delta_g + k^2)\psi = 0, \quad (x,z) \in \mathcal{S} \qquad \partial_z\psi(x,-1) = 0, \quad \psi(x,0) = u(x)\,, \tag{3.10}$$

where $\mathcal{S}$ is the strip $\mathcal{S} = \mathbb{R} \times (-1, 0)$. By using (3.8), the map $G_{\varepsilon,k}$ can be expressed in terms of ψ_k^u as

$$G_{\varepsilon,k}(u)(x) = -\partial_x \eta_\varepsilon(x) \partial_x \psi_k^u(x, 0) + \frac{1 + (\partial_x \eta_\varepsilon(x))^2}{1 + \eta_\varepsilon(x)} \partial_z \psi_k^u(x, 0)\,. \tag{3.11}$$

Therefore, using the Green formula together with (3.9) and the equation solved by ψ_k^u, we obtain that for $u, v \in H^{\frac{1}{2}}(\mathbb{R})$,

$$(G_{\varepsilon,k}(u), v) = \int_{\mathcal{S}} \Big(g^{-1} \nabla_{x,z} \psi_k^u \cdot \overline{\nabla_{x,z} \mathbf{v}} + k^2 \psi_k^u \overline{\mathbf{v}} \Big) (\det\, g)^{\frac{1}{2}} dx dz, \tag{3.12}$$

where $\mathbf{v}$ can be any H^1 function on $\mathcal{S}$ such that $\mathbf{v}(x, 0) = v(x)$.

To estimate the solution ψ_k^u of (3.10), we shall use the decomposition

$$\psi_k^u = u_k^H + u_k^r, \tag{3.13}$$

where u_k^H is the solution of

$$\big(- \Delta_{x,z} + k^2 \big) u_k^H = 0, \quad (x, z) \in \mathcal{S}, \quad \partial_z u_k^H(x, -1) = 0, \quad u_k^H(x, 0) = u(x), \tag{3.14}$$

$\mathcal{S}$ being again the strip $\mathbb{R} \times (-1, 0)$, and thus the remainder u_k^r is the solution of the elliptic problem with homogeneous boundary condition

$$\big(- \Delta_g + k^2 \big) u_k^r = \big(\Delta_g - k^2 \big) u_k^H, \quad (x, z) \in \mathcal{S}, \quad \partial_z u_k^r(x, -1) = 0, \quad u_k^r(x, 0) = 0. \tag{3.15}$$

By solving an ODE, one can write down explicitly the expression of the Fourier transform in x, $\hat{u}_k^H$ of u_k^H. We have:

$$\hat{u}_k^H(\xi, z) = \frac{\cosh\big(\sqrt{\xi^2 + k^2}(z + 1)\big)}{\cosh \sqrt{\xi^2 + k^2}} \hat{u}(\xi), \quad \xi \in \mathbb{R},\, z \in (-1, 0). \tag{3.16}$$

The estimate of ψ_k^u will be a consequence of the two following lemmas.

Lemma 3.2. *There exists $C > 0$ such that for every $k \in \mathbb{R}$, every $s \in \mathbb{R}$, every $u \in H^\infty(\mathbb{R})$,*

$$\|\Lambda^s u_k^H\|_{L^2(\mathcal{S})} \le C |\Lambda^s \Lambda_k^{-\frac{1}{2}} u|_{L^2}, \tag{3.17}$$

$$\|\Lambda^s \partial_z u_k^H\|_{L^2(\mathcal{S})} \le C |\Lambda^s \sqrt{D_x^2 + k^2} \Lambda_k^{-\frac{1}{2}} u|_{L^2}\,, \tag{3.18}$$

where Λ and Λ_k are the Fourier multipliers

$$\Lambda = (1 + D_x^2)^{\frac{1}{2}}, \quad \Lambda_k = (1 + k^2 + D_x^2)^{\frac{1}{2}}.$$

Proof of Lemma 3.2. First, we observe that it suffices to consider the case $s = 0$. Next, we note that there exists $C > 0$ such that for every $\omega \ge 0$, we have the inequalities

$$\int_{-1}^0 \frac{\cosh^2(\omega(z + 1))}{\cosh^2(\omega)}\, dz \le \frac{C}{1 + \omega}, \quad \int_{-1}^0 \frac{\sinh^2(\omega(z + 1))}{\cosh^2(\omega)}\, dz \le \frac{C}{1 + \omega}\,. \tag{3.19}$$

Indeed, inequalities (3.19) can be easily obtained for instance by performing the change of variable $z' = (1 + z)\omega$. Now (3.17) and (3.18) follow from (3.19) with $\omega = \sqrt{\xi^2 + k^2}$ via an application of the Parseval identity. This completes the proof of Lemma 3.2. $\square$

Let us now give the needed estimates for u_k^r.

Lemma 3.3. *Let us fix an integer $s \geq 0$. There exists $C > 0$ such that for every $k \in \mathbb{R}$, every $u \in H^\infty(\mathbb{R})$, the solution of* (3.15) *satisfies the estimate*

$$\|\Lambda^s \nabla_{x,z} u_k^r\|^2_{L^2(\mathcal{S})} + k^2 \|\Lambda^s u_k^r\|^2_{L^2(\mathcal{S})} \leq C \big|\Lambda^s \frac{\sqrt{D_x^2+k^2}}{\big(1+\sqrt{D_x^2+k^2}\big)^{\frac12}} u\big|^2_{L^2}. \tag{3.20}$$

Remark 3.4. *By a standard density argument, the statement of Lemma 3.2 and Lemma 3.3 may be extended to functional classes such that the right hand-side of the corresponding inequalities makes sense.*

Proof of Lemma 3.3. We have the following estimates

$$\|\Lambda^s \nabla_{x,z} u_k^r\|^2_{L^2(\mathcal{S})} + k^2 \|\Lambda^s u_k^r\|^2_{L^2(\mathcal{S})} \leq C\big(\|\Lambda^s \nabla_{x,z} u_k^H\|^2_{L^2} + k^2 \|\Lambda^s u_k^H\|^2_{L^2}\big). \tag{3.21}$$

Indeed, (3.21) for $s = 0$, is just the standard energy estimate: it suffices to take the L^2 scalar product of equation (3.15) with u_k^r and to perform integration by parts by using that u^r satisfies homogeneous boundary conditions. For $s \geq 1$ one may apply the standard argument for propagation of higher regularity in linear elliptic equations. Using (3.21) and Lemma 3.2 yield (3.20). This completes the proof of Lemma 3.3. $\square$

We are now in position to get (3.3). Thanks to (3.12), we have

$$\big(G_{\varepsilon,k} u, v\big) = \int_{\mathcal{S}} \Big(g^{-1} \nabla_{x,z} \psi_k^u \cdot \overline{\nabla_{x,z} \psi_k^v} + k^2 \psi_k^u \, \overline{\psi_k^v}\Big) (\det\, g)^{\frac12} dx dz. \tag{3.22}$$

Consequently, (3.3) follows by using the Cauchy-Schwarz inequality and (3.13), (3.17) (with $s = 0, 1$), (3.18) (with $s = 0$) and (3.20) (with $s = 0$).

As in [3], (3.4) will be a consequence of the trace formula. Let us choose $\chi(z)$ a smooth compactly supported cut-off function such that $\chi(0) = 1$ and χ is supported in $(-1, 1)$. We shall consider $\psi(x,z) = \chi(z)\psi_k^u(x,z)$. Note that since χ does not depend on x, we have $\hat{\psi}(\xi,z) = \chi(z)\hat{\psi}_k^u(x,z)$. We can write

$$\begin{aligned} |\hat{u}(\xi)|^2 = |\hat{\psi}(\xi,0)|^2 &\leq 2\int_{-1}^0 |\hat{\psi}(\xi,z)|\,|\partial_z \hat{\psi}(\xi,z)|\, dz \\ &\leq C \int_{-1}^0 \Big(|\hat{\psi}_k^u(\xi,z)|^2 + |\partial_z \hat{\psi}_k^u(\xi,z)||\hat{\psi}_k^u(\xi,z)|\Big) dz. \end{aligned}$$

This yields

$$\frac{\xi^2+k^2}{1+\sqrt{\xi^2+k^2}} |\hat{u}(\xi)|^2 \leq C \int_{-1}^0 \Big((\xi^2+k^2)|\hat{\psi}_k^u(\xi,z)|^2 + |\partial_z \hat{\psi}_k^u(\xi,z)|^2\Big) dz.$$

Consequently, we can integrate in ξ, use the Parseval identity and (3.22) to get

$$\Big|\frac{\sqrt{D_x^2+k^2}}{\big(1+\sqrt{D_x^2+k^2}\big)^{\frac12}} u\Big|^2_{L^2} \leq C \int_{\mathcal{S}} \Big(g^{-1}\nabla_{x,z}\psi_k^u \cdot \overline{\nabla_{x,z}\psi_k^u} + k^2 |\psi_k^u|^2\Big)(\det\, g)^{\frac12} dx dz = \big(G_{\varepsilon,k} u, u\big).$$

This ends the proof of (3.4). The proof of Proposition 3.1 is completed. $\square$

We next establish some additional qualitative properties of $G_{\varepsilon,k}$.

Proposition 3.5. *The operator $G_{\varepsilon,k}$ verifies:*

i) *For every $k \in \mathbb{R}$, $G_{\varepsilon,k} \in \mathcal{B}(H^s, H^{s-1})$ for every $s \in \mathbb{R}$.*
ii) *$G_{\varepsilon,k}$ depends continuously on k for $k \in \mathbb{R}$ and analytically on k for $k \in \mathbb{R}\backslash\{0\}$ in the operator norm of $\mathcal{B}(H^1, L^2)$.*

iii) *For every k, we have the decomposition*

$$G_{\varepsilon,k} = |D_x| + G^0_{\varepsilon,k}(x, D_x) \tag{3.23}$$

where for every k, $G^0_{\varepsilon,k}$ is a bounded operator on H^s, $G^0_{\varepsilon,k} \in \mathcal{B}(H^s, H^s)$ for every s.

Note that in this lemma we state mostly qualitative properties of $G_{\varepsilon,k}$ which hold locally in k. An immediate corollary of (3.23) is that $G_{\varepsilon,k}$ verifies an elliptic regularity criterion.

Corollary 3.6. *If $u \in H^s$ is such that $G_{\varepsilon,k}u \in H^s$, then $u \in H^{s+1}$.*

Proof of Proposition 3.5. We first prove i). By using (3.12), we get

$$\begin{aligned}\big(\Lambda^{s-\frac{1}{2}}G_{\varepsilon,k}(u), v\big) &= \big(G_{\varepsilon,k}(u), \Lambda^{s-\frac{1}{2}}v\big) \\ &= \int_{\mathcal{S}}\Big(g^{-1}\nabla_{x,z}\psi^u_k \cdot \overline{\nabla_{x,z}(\Lambda^{s-\frac{1}{2}}v^H_k)} + k^2\psi^u_k\overline{\Lambda^{s-\frac{1}{2}}v^H_k}\Big)(\det\, g)^{\frac{1}{2}}dxdz,\end{aligned} \tag{3.24}$$

where v^H_k is defined by

$$v^H_k(x,z) = \frac{\cosh\big(\sqrt{D_x^2 + k^2}(z+1)\big)}{\cosh\sqrt{D_x^2+k^2}}(v).$$

Next, we write

$$\begin{aligned}\big(\Lambda^{s-\frac{1}{2}}G_{\varepsilon,k}(u), v\big) = &\int_{\mathcal{S}}\Big(g^{-1}\nabla_{x,z}\Lambda^s\psi^u_k \cdot \overline{\nabla_{x,z}(\Lambda^{-\frac{1}{2}}v^H_k)} + k^2\Lambda^s\psi^u_k\overline{\Lambda^{-\frac{1}{2}}v^H_k}\Big)(\det\, g)^{\frac{1}{2}}dxdz + \\ &\int_{\mathcal{S}}\Big([\Lambda^s, (\det\, g)^{\frac{1}{2}}g^{-1}]\nabla_{x,z}\psi^u_k \cdot \overline{\nabla_{x,z}(\Lambda^{-\frac{1}{2}}v^H_k)} + k^2[\Lambda^s, (\det\, g)^{\frac{1}{2}}]\psi^u_k\overline{\Lambda^{-\frac{1}{2}}v^H_k}\Big)dxdz.\end{aligned}$$

For $s \geq 0$, an integer, we can apply the Cauchy-Schwarz inequality, Lemma 3.2 and Lemma 3.3 to get the bound

$$\big|\big(\Lambda^{s-\frac{1}{2}}G_{\varepsilon,k}u, v\big)\big| \leq C(k)|\Lambda^{s+\frac{1}{2}}u|_{L^2}\,|v|_{L^2}$$

and hence

$$|G_{\varepsilon,k}u|_{H^{s-\frac{1}{2}}} \leq C(k)|u|_{H^{s+\frac{1}{2}}}$$

(note that for this estimate we do not need to express precisely the dependence of $C(k)$ in k). Therefore $G_{\varepsilon,k}$ is continuous from $H^{s+\frac{1}{2}}$ to $H^{s-\frac{1}{2}}$ for $s \geq 0$ an integer. By interpolation $G_{\varepsilon,k}$ is continuous from H^s to H^{s-1} for $s \geq 1/2$. Next, since $G_{\varepsilon,k}$ is symmetric, by duality $G_{\varepsilon,k}$ is continuous from H^{1-s} to H^{-s} for $s \geq 1/2$. Thus $G_{\varepsilon,k}$ is continuous from H^s to H^{s-1} for every $s \in \mathbb{R}$.

Let us turn to the proof of ii). We shall first establish the continuity at zero. To study $\big(G_{\varepsilon,k} - G_{\varepsilon,0}\big)u$, we consider again ψ^u_k the solution of (3.10) and we shall use the expression

$$G_{\varepsilon,k}(u)(x) = -\partial_x\eta_\varepsilon(x)\partial_x\psi^u_k(x,0) + \frac{1 + (\partial_x\eta_\varepsilon(x))^2}{1+\eta_\varepsilon(x)}\partial_z\psi^u_k(x,0)\,. \tag{3.25}$$

We first notice that $\psi^u_k - \psi^u_0$ solves the elliptic equation

$$\Delta_g(\psi^u_k - \psi^u_0) = k^2\psi^u_k \tag{3.26}$$

with a homogeneous Dirichlet boundary condition on the upper boundary

$$\big(\psi^u_k - \psi^u_0\big)(x,0) = 0.$$

Note that this implies by the Poincare inequality that

$$\|\psi^u_k - \psi^u_0\|_{L^2(\mathcal{S})} \leq C\|\nabla(\psi^u_k - \psi^u_0)\|_{L^2(\mathcal{S})}.$$

Consequently, from the elliptic regularity, we get from (3.26) that

$$\|\psi_k^u - \psi_0^u\|_{H^2(\mathcal{S})} \le Ck^2 \|\psi_k^u\|_{L^2}.$$

Therefore, the trace theorem and the definition (3.25) yield

$$|\big(G_{\varepsilon,k} - G_{\varepsilon,0}\big)u|_{L^2} \le C\|\psi_k^u - \psi_0^u\|_{H^2(\mathcal{S})} \le Ck^2\|\psi_k^u\|_{L^2(\mathcal{S})}.$$

By using the Poincaré inequality and Lemma 3.2 and Lemma 3.3, we get

$$\|\psi_k^u\|_{L^2(\mathcal{S})} \le C\|\nabla\psi_k^u\|_{L^2(\mathcal{S})} \le C|\Lambda_k^{-1}(D_x^2+k^2)^{\frac{1}{2}}u|_{L^2}.$$

Therefore, we obtain that for $|k| \le 1$,

$$\|\psi_k^u\|_{L^2} \le C|u|_{H^{\frac{1}{2}}}.$$

Consequently, we get that there exists $C > 0$ such that for every $|k| \le 1$,

$$|\big(G_{\varepsilon,k} - G_{\varepsilon,0}\big)u|_{L^2} \le Ck^2|u|_{H^{\frac{1}{2}}}$$

which proves the continuity of $G_{\varepsilon,k}$ at zero as an operator in $\mathcal{B}(H^{\frac{1}{2}}, L^2)$ which is even better than the claimed property.

To prove the analyticity, it suffices to use again the decomposition (3.13) for ψ_k^u. From the explicit expression, of u_k^H, we get that it depends analytically on k for $k \neq 0$. Then u_k^r also depends analytically on k since u_k^r can be expressed as

$$u_r^k = R_g(k^2)F(k)\cdot(u_k^H)$$

where $R_g(\lambda) = (\Delta_g - \lambda)^{-1}$ is the resolvent of the Laplace Beltrami (with mixed boundary conditions) operator Δ_g and $F(k)$ is a linear operator depending on u_k^H. Since F and R_g depend analytically on k for $k \neq 0$, the result follows. More precisely, from the above considerations and very crude estimates, it follows immediately that $G_{\varepsilon,k}$ depends analytically on k in the operator norm $\mathcal{B}(H^{\frac{5}{2}}, L^2)$. From the Cauchy formula and the fact that $G_{\varepsilon,k}$ belongs to $\mathcal{B}(H^1, L^2)$ this yields the analyticity of $G_{\varepsilon,k}$ as an operator in $\mathcal{B}(H^1, L^2)$.

The proof of iii) which is for example detailed in [24] where moreover one tracks the dependence of the estimates on the regularity of the surface (see also [35]) relies on the construction of a parametrix for the elliptic equation (3.10). Let us just give the main steps in the argument. We define the operator $\Delta_g^{ap} = \Delta_g^{ap}(x, z, D_x, D_z)$ as

$$\Delta_g^{ap} \equiv -a\Big(\partial_z + \frac{b\partial_x + (1+\eta_\varepsilon)^{-1}\langle D_x\rangle}{a}\Big)\Big(\partial_z + \frac{b\partial_x - (1+\eta_\varepsilon)^{-1}\langle D_x\rangle}{a}\Big),$$

where

$$a = a(x,z) \equiv \frac{1 + (z+1)^2(\partial_x\eta_\varepsilon(x))^2}{(1+\eta_\varepsilon(x))^2}, \quad b = b(x,z) \equiv -\frac{\partial_x\eta_\varepsilon(x)(z+1)}{1+\eta_\varepsilon(x)}\,.$$

Using some basic pseudo-differential calculus, one can show that Δ_g^{ap} is a good approximation of $-\Delta_g + k^2$ in the sense that

$$\|(-\Delta_g + k^2) - \Delta_g^{ap}\|_{H^{s+1}(\mathcal{S})\to H^s(\mathcal{S})} \le C_{s,k}\,. \tag{3.27}$$

If we set $\eta_\pm(x, z, D_x) \equiv a^{-1}(-b\partial_x \pm (1+\eta_\varepsilon)^{-1}\langle D_x\rangle)$ then

$$\Delta_g^{ap} = -a(\partial_z - \eta_-(x,z,D_x))(\partial_z - \eta_+(x,z,D_x))\,.$$

We next find a parametrix $\phi_{ap} = \phi_{ap}(x, z, D_x)$ for Δ_g^{ap} such that $(\phi_{ap})_{/z=0} = Id$. This is given by

$$\phi_{ap} = \exp\big(-\int_z^0 \eta_+(x, z', D_x)dz'\big), \quad z \in [-1, 0]\,.$$

This linear operator ϕ_{ap} enjoys heat-flow type smoothing effects. Finally we have on the one hand

$$G_{\varepsilon,k}(u)(x) = -\partial_x\eta_\varepsilon(x)\partial_x\psi_k^u(x,0) + \frac{1+(\partial_x\eta_\varepsilon(x))^2}{1+\eta_\varepsilon(x)}\partial_z\psi_k^u(x,0)$$

and on the other hand that

$$\langle D_x\rangle(u) = -\partial_x\eta_\varepsilon(x)\partial_x\phi_{ap}(u)(x,0) + \frac{1+(\partial_x\eta_\varepsilon(x))^2}{1+\eta_\varepsilon(x)}\partial_z\phi_{ap}(u)(x,0)\,.$$

The result thus follows by proving the bound

$$|\nabla_{x,z}(\psi_k^u - \phi_{ap}(u))(x,0)|_{H^s(\mathbb{R})} \leq C_{k,s}|u|_{H^s(\mathbb{R})} \tag{3.28}$$

which is a consequence of the properties of ϕ_{ap} and elliptic regularity for the problem solved by $\psi_k^u - \phi_{ap}(u)$. This completes the proof of Proposition 3.5. □

Remark 3.7. Using the arguments of [35, Chapter 7.12], one may show that

$$G[\eta_\varepsilon] - \sqrt{|D_x|^2 + |D_y|^2(1+(\partial_x\eta_\varepsilon(x))^2)}$$

is a zero order pseudo-differential operator, independent of y, and thus its symbol $q(x,\xi_1,\xi_2)$ satisfies

$$\big|\partial_x^\alpha\partial_{\xi_1}^{\beta_1}\partial_{\xi_2}^{\beta_2}q(x,\xi_1,\xi_2)\big| \leq C_{\alpha,\beta_1,\beta_2}\langle|\xi_1|+|\xi_2|\rangle^{-\beta_1-\beta_2} \leq C_{\alpha,\beta_1,\beta_2}\langle\xi_1\rangle^{-\beta_1-\beta_2}\,. \tag{3.29}$$

Thus part iii) Proposition 3.5 is also a consequence of (3.29) with $\beta_2 = 0$ and the L^2 boundedness criterion for zero order pseudo-differential operators.

In the next proposition, we give useful commutator estimates.

Proposition 3.8 (Commutators)**.** *We have the following properties:*

i) *For every $s \geq 1$, $K > 0$, there exists $C_{s,K} > 0$ such that for every $u \in H^{s+\frac{1}{2}}$,*

$$\big|\big[\partial_x^s, G_{\varepsilon,k}\big]u\big|_{H^{\frac{1}{2}}} \leq C_{s,K}\Big(\Big|\frac{|D_x|}{1+|D_x|^{\frac{1}{2}}}u\Big|_{H^s} + |k|\,|u|_{H^s}\Big), \quad \forall k,\ |k| \leq K. \tag{3.30}$$

ii) *For every $K > 0$ and every smooth function $f(x) \in \mathcal{S}(\mathbb{R})$, there exists $C_K > 0$ such that for every $u \in H^{\frac{1}{2}}$,*

$$\Big|\mathrm{Re}\Big(f\partial_x u, G_{\varepsilon,k}u\Big)\Big| \leq C_K\Big(\Big|\frac{|D_x|}{1+|D_x|^{\frac{1}{2}}}u\Big|_{L^2} + |k|\,|u|_{L^2}\Big), \quad \forall k,\ |k| \leq K. \tag{3.31}$$

Note that in this proposition we do not pay attention to the dependence of the estimates in k for large k, since this will not be needed.

Proof of Proposition 3.8. To prove i), we shall estimate

$$I = \big(\Lambda^{\frac{1}{2}}\partial_x^s G_{\varepsilon,k}u, v\big) - \big(\Lambda^{\frac{1}{2}}G_{\varepsilon,k}\partial_x^s u, v\big).$$

By using again (3.12), we write

$$\begin{aligned}
I &= (-1)^s\int_{\mathcal{S}}\Big(g^{-1}\nabla_{x,z}\psi_k^u\cdot\overline{\nabla_{x,z}\Lambda^{\frac{1}{2}}\partial_x^s v_k^H} + k^2\psi_k^u\,\overline{\Lambda^{\frac{1}{2}}\partial_x^s v_k^H}\Big)(\det g)^{\frac{1}{2}}\,dxdz\\
&\quad - \int_{\mathcal{S}}\Big(g^{-1}\nabla_{x,z}\psi_k^{\partial_x^s u}\cdot\overline{\nabla_{x,z}\Lambda^{\frac{1}{2}}v_k^H} + k^2\psi_k^{\partial_x^s u}\,\overline{\Lambda^{\frac{1}{2}}v_k^H}\Big)(\det g)^{\frac{1}{2}}\,dxdz\\
&= \int_{\mathcal{S}}\Big(g^{-1}\big(\nabla_{x,z}\partial_x^s\psi_k^u - \nabla_{x,z}\psi_k^{\partial_x^s u}\big)\cdot\overline{\nabla_{x,z}\Lambda^{\frac{1}{2}}v_k^H} + k^2\big(\partial_x^s\psi_k^u - \psi_k^{\partial_x^s u}\big)\overline{\Lambda^{\frac{1}{2}}v_k^H}\Big)(\det g)^{\frac{1}{2}}\,dxdz\\
&\quad + \int_{\mathcal{S}}\Big(\big[\partial_x^s, (\det g)^{\frac{1}{2}}g^{-1}\big]\nabla_{x,z}\psi_k^u\cdot\overline{\nabla_{x,z}\Lambda^{\frac{1}{2}}v_k^H} + k^2\big([\partial_x^s, (\det g)^{\frac{1}{2}}]\psi_k^u\,\overline{\Lambda^{\frac{1}{2}}v_k^H}\Big)\,dxdz\\
&\equiv J_1 + J_2.
\end{aligned}$$

We estimate the second integral above by

$$|J_2| \leq C_K\Big(\|\Lambda^{\frac12} v_k^H\|_{L^2(\mathcal{S})} \big\|[\partial_x^s,(\det g)^{\frac12} g^{-1}]\nabla_{x,z}\psi_k^u\big\|_{H^1(\mathcal{S})} + |k|\|\Lambda^{\frac12} v_k^H\|_{L^2(\mathcal{S})} \big\|[\partial_x^s,(\det g)^{\frac12}]\psi_k^u\big\|_{L^2(\mathcal{S})}\Big)$$

and hence, by using again Lemma 3.2, Lemma 3.3 and standard commutator estimates, we find

$$|J_2| \leq C_{s,K}|v|_{L^2}\Big(\Big|\frac{|D_x|}{1+|D_x|^{\frac12}}u\Big|_{H^s} + |k|\,|u|_{H^s}\Big).$$

In a similar way, we estimate J_1 as follows

$$|J_1| \leq C_{s,K}|v|_{L^2}\Big(\big\|\nabla_{x,z}\partial_x^s\psi_k^u - \nabla_{x,z}\psi_k^{\partial_x^s u}\big\|_{H^1(\mathcal{S})} + |k|\big\|\partial_x^s\psi_k^u - \psi_k^{\partial_x^s u}\big\|_{L^2(\mathcal{S})}\Big).$$

To conclude, we notice that $\psi = \partial_x^s\psi_k^u - \psi_k^{\partial_x^s u}$ solves the elliptic equation

$$-\Delta_g\psi + k^2\psi = [\partial_x^s,\Delta_g]\psi_k^u \tag{3.32}$$

with the homogeneous boundary conditions

$$\partial_z\psi(x,-1)=0,\quad \psi(x,0)=0.$$

Consequently, the H^s elliptic regularity estimates for (3.32), the Poincaré inequality and again Lemma 3.2 and Lemma 3.3 yield

$$|J_1| \leq C_{s,K}|v|_{L^2}\Big(\Big|\frac{|D_x|}{1+|D_x|^{\frac12}}u\Big|_{H^s} + |k|\,|u|_{H^s}\Big).$$

This ends the proof of i).

Let us now prove ii). We use again (3.12) to write

$$\begin{aligned}
(G_{\varepsilon,k}u, f\partial_x u) &= \int_{\mathcal{S}}\Big(g^{-1}\nabla\psi_k^u\cdot\overline{\nabla(f\partial_x\psi_k^u)} + k^2\psi_k^u\overline{f\partial_x\psi_k^u}\Big)(\det g)^{\frac12}\,dxdz\\
&= \int_{\mathcal{S}}\Big(g^{-1}\nabla\psi_k^u\cdot\overline{\partial_x\nabla\psi_k^u}\Big)\overline{f}(\det g)^{\frac12}\,dxdz\\
&\quad+\int_{\mathcal{S}}\Big(g^{-1}\nabla\psi_k^u\cdot\overline{\partial_x\psi_k^u\,\nabla f} + k^2\psi_k^u\overline{f\partial_x\psi_k^u}\Big)(\det g)^{\frac12}\,dxdz\\
&\equiv I_1 + I_2.
\end{aligned}$$

By using again Lemma 3.2 and Lemma 3.3, we immediately get that

$$|I_2| \leq C_{s,K}\Big(\Big|\frac{|D_x|}{1+|D_x|^{\frac12}}u\Big|_{L^2}^2 + |k|^2|u|_{L^2}^2\Big).$$

To estimate I_1, we first integrate by parts to obtain

$$2\,\mathrm{Re}\, I_1 = -\int_{\mathcal{S}}\partial_x\big(\overline{f}\,(\det g)^{\frac12}g^{-1}\big)\nabla\psi_k^u\cdot\overline{\nabla\psi_k^u}\,dxdz$$

and then by Lemma 3.2 and Lemma 3.3, we also get

$$|\mathrm{Re}\, I_1| \leq C_{s,K}\Big(\Big|\frac{|D_x|}{1+|D_x|^{\frac12}}u\Big|_{L^2}^2 + |k|^2|u|_{L^2}^2\Big).$$

This yields the desired estimate for $\mathrm{Re}\,\big(G_{\varepsilon,k}u, f\partial_x u\big)$.

This ends the proof of Proposition 3.8. □

Let us set

$$G_k[\eta]u = e^{-iky}G[\eta](ue^{iky})$$

for functions $\eta(x)$, $u(x)$ which depends on x only. We are interested in estimates of $D^j_\eta G_k[\eta_\varepsilon]u \cdot \big(h_1, \dots, h_j\big)$. We shall use the notation

$$D^j_\eta G_{\varepsilon,k} u \cdot \big(h_1, \dots, h_j\big) = D^j_\eta G_k[\eta_\varepsilon]u \cdot \big(h_1, \dots, h_j\big).$$

Proposition 3.9. *For every $s > 1/2$, we have the estimate*

$$\Big|D^j_\eta G_{\varepsilon,k} u \cdot \big(h_1, \dots, h_j\big)\Big|_{H^{s-\frac{1}{2}}} \leq C_s \Big|\frac{\sqrt{D_x^2 + k^2}}{\big(1 + \sqrt{D_x^2+k^2}\big)^{\frac{1}{2}}}u\Big|_{H^s} \prod_{i=1}^{j} |h_i|_{H^{s+1}}. \tag{3.33}$$

Proof. It suffices to take the derivative of (3.24) with respect to η and then to use the standard Sobolev-Gagliardo-Nirenberg-Moser estimates for products in Sobolev spaces and again Lemma 3.2 and Lemma 3.3. This completes the proof of Proposition 3.9. $\square$

4. Study of the operator $L(k)$ arising in the linearization of the Hamiltonian

As a preliminary, we first establish the following statement.

Lemma 4.1. *$L(k)$ has a self-adjoint realization on $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ with domain $H^2(\mathbb{R}) \times H^1(\mathbb{R})$.*

Proof. We first notice that $L(k)$ enjoys an elliptic regularity property, namely if $u = (u_1, u_2) \in L^2 \times L^2$ is such that $L(k)u \in L^2 \times L^2$ then $u \in H^2 \times H^1$. Indeed, using the first equation and the elliptic regularity for the second order operator $P_{\varepsilon,k}$, we obtain that $u_1 \in H^1$. Then, using the elliptic regularity for $G_{\varepsilon,k}$ established in Corollary 3.6 and the second equation, we obtain that $u_2 \in H^1$. Finally, using again the elliptic regularity for $P_{\varepsilon,k}$, we obtain that $u_1 \in H^2$.

Next, we also observe that $L(k)$ is symmetric in $H^\infty \times H^\infty$, namely

$$(L(k)u, v) = (u, L(k)v), \quad \forall\, u, v \in H^\infty \times H^\infty\,. \tag{4.1}$$

Moreover, let us consider the closure $\overline{L(k)}$ of $L(k)$ defined on the domain

$$D(\overline{L(k)}) = \{u \in L^2 \times L^2 \,:\, \exists\, u_n \in H^\infty \times H^\infty, u_n \to u \text{ in } L^2 \times L^2,\ L(k)u_n \text{ converges in } L^2 \times L^2\}.$$

We shall show that $\overline{L(k)}$ is self adjoint and that $D(\overline{L(k)}) = H^2 \times H^1$. By definition, the adjoint of $\overline{L(k)}$, denoted by $\overline{L(k)}^\star$ has the domain

$$D(\overline{L(k)}^\star) = \{u \in L^2 \times L^2 \,:\, \exists\, C > 0,\ |(u, L(k)v)| \leq C\|v\|_{L^2 \times L^2},\ \forall\, v \in D(\overline{L(k)})\}$$

and moreover, the following inclusions hold:

$$H^2 \times H^1 \subset D(\overline{L(k)}) \subset D(\overline{L(k)}^\star) \subset H^2 \times H^1\,.$$

Indeed, the first inclusion follows from the density of $H^\infty \times H^\infty$ in $H^2 \times H^1$ and the fact that $L(k)$ is continuous from $H^2 \times H^1$ to $L^2 \times L^2$. The second inclusion follows from the fact that $L(k)$ is symmetric (see (4.1)). The third inclusion is the most difficult to check. It follows from the elliptic regularity, since $D(\overline{L(k)}^\star)$ can be also seen as the function in $L^2 \times L^2$ such that $L(k)u$ (a priori defined in a weak sense) belongs to $L^2 \times L^2$. This completes the proof of Lemma 4.1. $\square$

4.1. **Essential spectrum.** Our aim is now to locate the essential spectrum of $L(k)$.

Proposition 4.2. *For every $\varepsilon \in (0, \varepsilon_0]$, and for every $k \in \mathbb{R}$, there exists $c_k \geq 0$, such that*

$$\sigma_{ess}(L(k)) \subset [c_k, +\infty) \subset [0, +\infty).$$

Moreover, for $k \neq 0$, we have $c_k > 0$.

Note that for $k \neq 0$ the essential spectrum of $L(k)$ is included in $(0, +\infty)$.

Proof of Proposition 4.2. Since $L(k)$ is self-adjoint, its spectrum is real. We thus only have to prove that $\gamma + L(k)$ is Fredholm with zero index for $\gamma \geq 0$, $k \neq 0$ and for $\gamma > 0$, $k = 0$. Towards this, we shall prove that $L(k) + \gamma$ can be written as

$$L(k) + \gamma = \mathcal{I}(\gamma, k) + \mathcal{K}(\gamma, k),$$

where $\mathcal{I}(\gamma, k)$ is an invertible operator for $\gamma > 0$ with domain $H^2 \times H^1$ and $\mathcal{K}(\gamma, k)$ is a relatively compact perturbation.

Let us first have a look to the asymptotic behaviour of the coefficients of $L(k)$. In Theorem 1.2, we have already recalled that η_ε, $\partial_x \varphi_\varepsilon$ and their higher order derivatives have an exponential decay towards zero at infinity. Moreover, since for a solitary wave, we have $G[\eta_\varepsilon]\varphi_\varepsilon = -\partial_x \eta_\varepsilon$, we also get that $G[\eta_\varepsilon]\varphi_\varepsilon$ and its derivatives tend to zero exponentially fast at infinity. This yields in particular that v_ε and $\partial_x Z_\varepsilon$ have an exponential decay towards 0 when x tends to $\pm\infty$. Consequently, we can first write the decomposition

$$L(k) + \gamma = L_0(\gamma, k) + C(k),$$

where

$$L_0(\gamma, k) = \begin{pmatrix} -\beta\zeta^{-3}\partial_x^2 + \beta k^2 + \alpha + \gamma & -(1 - v_\varepsilon)\partial_x \\ \partial_x & \gamma + G_{\varepsilon,k} \end{pmatrix}$$

and

$$C(k) = \begin{pmatrix} 3\beta\zeta^{-4}\zeta'\partial_x - \beta k^2(1 - \zeta^{-1}) + (v_\varepsilon - 1)\partial_x Z_\varepsilon & 0 \\ -\partial_x(v_\varepsilon \cdot) & 0 \end{pmatrix}$$

where ζ is defined as

$$\zeta(x) = (1 + (\partial_x \eta_\varepsilon(x))^2)^{\frac{1}{2}}.$$

Note that the function $\zeta(x)$ has an exponential decay towards 1 when x tends to $\pm\infty$, while its derivatives decay exponentially to zero.

The domain of $L_0(\gamma, k)$ is again $H^2 \times H^1$. Consequently, thanks to the exponential decay in its coefficients, we get that $C(k)$ is a relatively compact perturbation i.e $C(k)$ seen as an operator in $\mathcal{B}(H^2 \times H^1, L^2 \times L^2)$ is compact.

Next, by using that there exists $\varepsilon_0 > 0$ such that for $\varepsilon \leq \varepsilon_0$ we have $1 - v_\varepsilon > 0$, we write the factorization

$$L_0(\gamma, k) = A_1 L_1(\gamma, k) + C_1(\gamma, k),$$

where

$$L_1(\gamma, k) = \begin{pmatrix} -\beta(1 - v_\varepsilon)^{-1}\zeta^{-3}\partial_x^2 + \beta k^2 + \alpha + \gamma & -\partial_x \\ \partial_x & \gamma + G_{\varepsilon,k} \end{pmatrix},$$

$$C_1(\gamma, k) = \begin{pmatrix} (\beta k^2 + \alpha + \gamma)v_\varepsilon & 0 \\ 0 & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 1 - v_\varepsilon & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that $C_1(\gamma, k)$ is again a relatively compact perturbation because of the decay of v_ε at infinity, while A_1 is just an invertible matrix. We can simplify $L_1(\gamma, k)$ a little bit by writing

$$L_1(\gamma, k) = L_2(\gamma, k)B_1 + C_2,$$

where

$$L_2(\gamma,k)=\begin{pmatrix} -\beta\partial_x^2+\beta k^2+\alpha+\gamma & -\partial_x \\ \partial_x & \gamma+G_{\varepsilon,k}\end{pmatrix},$$

$$B_1=\begin{pmatrix} \zeta^{-3}(1-v_\varepsilon)^{-1} & 0 \\ 0 & 1\end{pmatrix},$$

$$C_2=\begin{pmatrix} -\beta\big[(1-v_\varepsilon)^{-1}\zeta^{-3},\partial_x^2\big]-(\beta k^2+\alpha+\gamma)(1-\zeta^{-3}(1-v_\varepsilon)^{-1}) & 0 \\ \partial_x\Big(\big(1-\zeta^{-3}(1-v_\varepsilon)^{-1}\big)\cdot\Big) & 0\end{pmatrix}.$$

Again, we see that C_2 is a relatively compact perturbation since because of the decay of its coefficients it is compact as an operator in $\mathcal{B}(H^2\times H^1, L^2\times L^2)$. Moreover, B_1 is just an invertible matrix.

Next, to simplify the expression of L_2, we shall use a factorization inspired by the work of Mielke [26]. Thanks to (3.4), we note that the operator $\gamma+G_{\varepsilon,k}$ satisfies for some $c=c(k)>0$ the estimate:

$$\big((\gamma+G_{\varepsilon,k})u,u\big)\geq c\Big(\Big|\frac{|D_x|}{1+|D_x|^{\frac12}}u\Big|_{L^2}^2+(\gamma+k^2)|u|_{L^2}^2\Big).$$

Consequently, for $\gamma\geq 0$ and $k>0$ or for $\gamma=0$ and $k>0$ we have that for some $c>0$ (depending on γ and k),

$$\big((\gamma+G_{\varepsilon,k})u,u\big)\geq c|u|_{H^{\frac12}}^2. \tag{4.2}$$

Moreover, thanks to (3.3), we also have that

$$\big((\gamma+G_{\varepsilon,k})u,v\big)\leq C|u|_{H^{\frac12}}|v|_{H^{\frac12}}$$

thus the quadratic form $\big((\gamma+G_{\varepsilon,k})\cdot,\cdot\big)$ is continuous and coercive on $H^{\frac12}$. By the Lax-Milgram lemma and Corollary 3.6, we thus get that the operator $\gamma+G_{\varepsilon,k}$ defined on L^2 with domain H^1 is invertible for every (γ,k) such that $\gamma\geq 0$ and $k>0$ or $\gamma>0$ and $k=0$. The existence of $(\gamma+G_{\varepsilon,k})^{-1}$ allows to introduce the factorization

$$L_2(\gamma,k)=A_2(\gamma,k)L_3(\gamma,k)B_2(\gamma,k)$$

where

$$L_3(\gamma,k)=\begin{pmatrix} -\beta\partial_{xx}+\beta k^2+\alpha+\gamma+\partial_x(\gamma+G_{\varepsilon,k})^{-1}\partial_x & 0 \\ 0 & \gamma+G_{\varepsilon,k}\end{pmatrix},$$

$$A_2(\gamma,k)=\begin{pmatrix} 1 & -\partial_x(\gamma+G_{\varepsilon,k})^{-1} \\ 0 & 1\end{pmatrix},$$

$$B_2(\gamma,k)=\begin{pmatrix} 1 & 0 \\ (\gamma+G_{\varepsilon,k})^{-1}\partial_x & 1\end{pmatrix}=A_2(\gamma,k)^*.$$

Note that $A_2(\gamma,k)$ and $B_2(\gamma,k)$ are bounded invertible operators on $L^2\times L^2$. To get our last simplification, we shall prove that the operator

$$\partial_x(\gamma+G_{\varepsilon,k})^{-1}\partial_x-\partial_x(\gamma+G_{0,k})^{-1}\partial_x \tag{4.3}$$

is a compact operator in $\mathcal{B}(H^2\times H^1, L^2\times L^2)$, where the operator $G_{0,k}$ is the Dirichlet Neumann for the flat surface $\eta=0$,

$$G_{0,k}u=e^{-iky}G[0](ue^{iky}).$$

Coming back to (3.16), we obtain that $G_{0,k}$ is a Fourier multiplier, namely

$$G_{0,k}=\tanh(D_x^2+k^2)\sqrt{D_x^2+k^2}\,.$$

In order to study the compactness properties of (4.3), we shall use the following lemma.

Lemma 4.3. *The operator $R_\varepsilon = G_{\varepsilon,k} - G_{0,k}$ is a bounded operator in $\mathcal{B}(H^1, L^2)$ and a compact operator from H^s to L^2 for every $s > 1$.*

Let us postpone the proof of Lemma 4.3. We now show how we can end the proof of Proposition 4.2 by using Lemma 4.3. We write

$$\gamma + G_{\varepsilon,k} = (\mathrm{Id} + R_\varepsilon^{(1)})(\gamma + G_{0,k}), \quad R_\varepsilon^{(1)} = R_\varepsilon(\gamma + G_{0,k})^{-1}.$$

We have that $\mathrm{Id} + R_\varepsilon^{(1)}$ is a bounded invertible operator on L^2 and

$$(\mathrm{Id} + R_\varepsilon^{(1)})^{-1} = (\gamma + G_{0,k})(\gamma + G_{\varepsilon,k})^{-1}.$$

Moreover, thanks to Lemma 4.3 and since $(\gamma + G_{0,k})^{-1}$ is a bounded operator from H^s to H^{s+1}, we get that $R_\varepsilon^{(1)}$ is a compact operator from H^s to L^2 for every $s > 0$. Next, since

$$(\mathrm{Id} + R_\varepsilon^{(1)})^{-1} = \mathrm{Id} - (\mathrm{Id} + R_\varepsilon^{(1)})^{-1} R_\varepsilon^{(1)},$$

we get that $(\mathrm{Id} + R_\varepsilon^{(1)})^{-1} = \mathrm{Id} + R_\varepsilon^{(2)}$ where $R_\varepsilon^{(2)}$ is a compact operator from H^s to L^2 for every $s > 0$. To conclude, we write that

$$(\gamma + G_{\varepsilon,k})^{-1} = (\gamma + G_{0,k})^{-1}(\mathrm{Id} + R_\varepsilon^{(1)})^{-1} = (\gamma + G_{0,k})^{-1} + (\gamma + G_{0,k})^{-1} R_\varepsilon^{(2)}$$

which yields

$$\partial_x(\gamma + G_{\varepsilon,k})^{-1}\partial_x = \partial_x(\gamma + G_{0,k})^{-1}\partial_x + \partial_x(\gamma + G_{0,k})^{-1} R_\varepsilon^{(2)}\partial_x$$

and we observe that $\partial_x(\gamma + G_{0,k})^{-1}$ is a bounded operator on L^2, that ∂_x is a bounded operator from H^2 to H^1 and that $R_\varepsilon^{(2)}$ is a compact operator from H^1 to L^2. Consequently, we have obtained that

$$\partial_x(\gamma + G_{\varepsilon,k})^{-1}\partial_x = \partial_x(\gamma + G_{0,k})^{-1}\partial_x + R_\varepsilon^{(3)},$$

where $R_\varepsilon^{(3)}$ is a compact operator from H^2 to L^2. This finally allows to write that

$$L_3(\gamma, k) = L_4(\gamma, k) + C_3(\gamma, k)$$

where

$$L_4(\gamma, k) = \begin{pmatrix} -\beta\partial_x^2 + \beta k^2 + \alpha + \gamma + \partial_x(\gamma + G_{0,k})^{-1}\partial_x & 0 \\ 0 & \gamma + G_{\varepsilon,k} \end{pmatrix}$$

and $C_3(\gamma, k)$ is a relatively compact perturbation.

Gathering all our transformations, we find that

$$\gamma + L(k) = A_1 A_2(\gamma, k) L_4(\gamma, k) B_2(\gamma, k) B_1 + \mathcal{K} \tag{4.4}$$

where $\mathcal{K}$ is a relatively compact perturbation for $\gamma \geq 0$, $k \neq 0$ and for $\gamma > 0$, $k = 0$. Consequently to get that $\gamma + L(k)$ is Fredholm with index zero, it suffices to prove that $A_1 A_2(\gamma, k) L_4(\gamma, k) B_2(\gamma, k) B_1$ is invertible. Since A_1, A_2, B_1, B_2 are bounded invertible operators, we only have to prove that $L_4(\gamma, k)$ is invertible. Moreover, we see that L_4 is a diagonal operator and we have already seen that $\gamma + G_{\varepsilon,k}$ is invertible for $\gamma \geq 0$, $k \neq 0$ or $\gamma > 0$, $k = 0$. Therefore, it only remains to study the invertibility of

$$-\beta\partial_x^2 + \beta k^2 + \alpha + \gamma + \partial_x(\gamma + G_{0,k})^{-1}\partial_x.$$

This operator is just the Fourier multiplier by

$$\begin{aligned} m(\xi) &= \beta\xi^2 + \beta k^2 + \alpha + \gamma - \frac{\xi^2}{\gamma + \tanh(\sqrt{\xi^2 + k^2})\sqrt{\xi^2 + k^2}} \\ &\geq \beta\xi^2 + \beta k^2 + \alpha + \gamma - \frac{|\xi|}{\tanh|\xi|}. \end{aligned}$$

As observed in [26], we have the following statement.

Lemma 4.4. *For $\beta > 1/3$, we have the inequality*

$$\beta x^2 - \frac{x}{\tanh(x)} \geq -1, \quad \forall\, x \geq 0.$$

Proof. Let us set $f(x) = \beta x^2 - \frac{x}{\tanh(x)}$, then $f'(x) = g(x)/(e^x - e^{-x})^2$, where

$$g(x) = 2\beta x(e^{2x} + e^{-2x}) - (e^{2x} - e^{-2x}) + (4 - 4\beta)x\,.$$

We have that near zero $g(x) = 8(\beta - 1/3)x^3 + \mathcal{O}(x^4)$ and that

$$g^{(4)}(x) = 32\beta x(e^{2x} + e^{-2x}) + 16(4\beta - 1)(e^{2x} - e^{-2x}) > 0, \quad \forall\, x \geq 0.$$

Hence $g^{(3)}(x)$ is an increasing function and since for $\beta > 1/3$, $g^{(3)}(0) > 0$, we obtain that $g^{(3)}(x) > 0$ for every $x \geq 0$. Next, in a similar way, we obtain successively that $g''(x)$, $g'(x)$ and $g(x)$ are non-negative. Therefore $f(x)$ is an increasing function which implies that for every $x \geq 0$, $f(x) \geq f(0) = -1$. This ends the proof of Lemma 4.4. □

Using Lemma 4.4, we get

$$m(\xi) \geq \beta k^2 + \gamma + \alpha - 1.$$

Since $\alpha = 1 + \varepsilon^2$, $m(\xi)$ is uniformly bounded from below by a positive number for $\gamma \geq 0$. Consequently, the operator L_4 is invertible. To end the proof of Proposition 4.2, it only remains to give the proof of Lemma 4.3 .

Proof of Lemma 4.3. Thanks to the fundamental theorem of calculus, we can write

$$(G_{\varepsilon,k} - G_{0,k})\varphi = e^{-iky} \int_0^1 \big(D_\eta G[s\eta_\varepsilon](e^{iky}\varphi) \cdot \eta_\varepsilon\big)\, ds.$$

Using Lemma 1.1, we have

$$(4.5)\quad \big(G_{\varepsilon,k} - G_{0,k}\big)\varphi = \int_0^1 \Big(- G_k[s\eta_\varepsilon](\eta_\varepsilon Z_k(s\eta_\varepsilon, \varphi)) - \partial_x\big(\eta_\varepsilon(\partial_x\varphi - sZ_k(s\eta_\varepsilon, \varphi)\partial_x\eta_\varepsilon)\big) + k^2\eta_\varepsilon\varphi\Big) ds,$$

where we use the notations

$$G_k[\eta]\varphi = e^{-iky}G[\eta](e^{iky}\varphi), \quad Z_k(\eta, \varphi) = \frac{G_k[\eta]\varphi + \partial_x\eta\partial_x\varphi}{1 + |\partial_x\eta|^2}.$$

In this formula, it seems at first sight that $\big(G_{\varepsilon,k} - G_{0,k}\big)$ is a second order operator. Nevertheless, by using iii) of Proposition 3.5, basic commutator estimates and the decay of η_ε and its derivatives, we get that

$$\begin{aligned}\big(G_{\varepsilon,k} - G_{0,k}\big)\varphi = \int_0^1 \Big(&\partial_x^2\varphi\big(\frac{\eta_\varepsilon}{1 + s^2(\partial_x\eta_\varepsilon)^2} - \eta_\varepsilon + \frac{\eta_\varepsilon s^2(\partial_x\eta_\varepsilon)^2}{1 + s^2(\partial_x\eta_\varepsilon)^2}\big) \\ &+ \partial_x|D_x|\varphi\big(\frac{s\eta_\varepsilon\partial_x\eta_\varepsilon}{1 + s^2(\partial_x\eta_\varepsilon)^2} - \frac{s\eta_\varepsilon\partial_x\eta_\varepsilon}{1 + s^2(\partial_x\eta_\varepsilon)^2}\big)\Big) ds + \mathcal{R} = \mathcal{R},\end{aligned}$$

where the operator $\mathcal{R}$ is a sum of terms which are all made of the product of a first order (pseudo differential) operator i.e. belonging to $\mathcal{B}(H^s, H^{s-1})$ and of a rapidly decreasing function. This yields that $\mathcal{R}$ is compact as an operator from H^s to L^2 for every $s > 1$ and ends the proof of Lemma 4.3. □

This also ends the proof of Proposition 4.2. □

4.2. **Negative eigenvalues of $L(k)$.** The next step is the study of the eigenvalues of $L(k)$ outside the essential spectrum.

As in [26], it is convenient to introduce a reduced operator in order to study the eigenvalues of $L(k)$. We first define the operator

$$Mu = -\partial_x^{-1} G_{\varepsilon,0} \partial_x^{-1} u$$

where ∂_x^{-1} is defined by the division by $i\xi$ in the Fourier space. Note that M is well-defined for smooth functions whose support of their Fourier transform does not meet zero. To study M, it is convenient to introduce the bilinear symmetric form

$$Q(u,v) = (Mu, v),$$

for $u, v \in H^\infty(\mathbb{R})$ such that $\hat{u}$, $\hat{v}$ have supports which do not meet zero.

We have the following statement:

Lemma 4.5. *Q extends to a continuous and coercive bilinear form on $H^{-\frac{1}{2}} \times H^{-\frac{1}{2}}$.*

As a consequence of this statement, we get thanks to the continuity of Q that M is well-defined as an operator in $\mathcal{B}(H^{-\frac{1}{2}}, H^{\frac{1}{2}})$. Moreover, thanks to the Lax-Milgram lemma, we can thus define the inverse M^{-1} as an operator in $\mathcal{B}(H^{\frac{1}{2}}, H^{-\frac{1}{2}})$. By using (3.23) in Proposition 3.5, we can then get that M is a continuous bijection from H^s to H^{s+1} for every $s \in \mathbb{R}$.

Proof of Lemma 4.5. We notice that

$$Q(u,v) = \big(G_{\varepsilon,0}\partial_x^{-1}u, \partial_x^{-1}v\big)$$

consequently, thanks to (3.3) in Proposition 3.1, we get that

$$|Q(u,v)| \leq C|u|_{H^{-\frac{1}{2}}}\, |v|_{H^{-\frac{1}{2}}}.$$

In a similar way, we get that

$$Q(u,u) \geq c|u|^2_{H^{-\frac{1}{2}}}$$

thanks to (3.4). This ends the proof of Lemma 4.5. $\square$

Next, as in [26], we can use the operators M and M^{-1} to notice that

$$\begin{aligned} \big(L(0)U, U\big) \quad = \quad & \Big(\big(-P_{\varepsilon,0} + \alpha - \gamma_\varepsilon \partial_x Z_\varepsilon\big)U_1, U_1\Big) - \big(M^{-1}(\gamma_\varepsilon U_1), \gamma_\varepsilon U_1\big) \\ & + \Big(M\big(\partial_x U_2 - M^{-1}(\gamma_\varepsilon U_1)\big), \partial_x U_2 - M^{-1}(\gamma_\varepsilon U_1)\Big) \end{aligned} \tag{4.6}$$

where we have set $\gamma_\varepsilon \equiv 1 - v_\varepsilon$. Since M is nonnegative, we obtain that

$$\big(L(0)U, U\big) \geq \Big(\big(-P_{\varepsilon,0} + \alpha - \gamma_\varepsilon \partial_x Z_\varepsilon\big)U_1, U_1\Big) - \big(M^{-1}(\gamma_\varepsilon U_1), \gamma_\varepsilon U_1\big) \equiv (A_\varepsilon U_1, U_1).$$

By using that for the solitary wave η_ε, we have $G[\eta_\varepsilon]\varphi_\varepsilon = -\partial_x \eta_\varepsilon$, we get that

$$\gamma_\varepsilon = \frac{1 - \partial_x \varphi_\varepsilon}{1 + (\partial_x \eta_\varepsilon)^2}, \quad Z_\varepsilon = -\gamma_\varepsilon \partial_x \eta_\varepsilon\,.$$

Therefore, we obtain the expression

$$A_\varepsilon \eta = -P_{\varepsilon,0}\eta + \alpha\eta + \gamma_\varepsilon \partial_x(\gamma_\varepsilon \partial_x \eta_\varepsilon)\eta - \gamma_\varepsilon M^{-1}(\gamma_\varepsilon \eta). \tag{4.7}$$

The spectrum of the operator A_ε is studied in [26]. Note that our notations are slightly different from the one of Mielke in [26], in particular, here α is the rescaled coefficient coming from the term taking into account the gravity in the equation. After a suitable rescaling A_ε tends (in a rather weak sense) to the operator obtained by linearizing the KdV equation about the KdV solitary wave. This allows to prove the following statement.

Proposition 4.6 (Mielke [26]). *There exists $\varepsilon_0 > 0$ such that for every ε, $\varepsilon \in (0, \varepsilon_0)$, the spectrum of A_ε which is a self adjoint operator on L^2 with domain H^2 consists of a negative simple eigenvalue λ_ε^-, the simple eigenvalue zero and the remaining of the spectrum is included in $[\lambda_\varepsilon, +\infty[$ for some $\lambda_\varepsilon > 0$.*

Proof. By using similar arguments as in the proof of Proposition 4.2, we can first locate the essential spectrum of A_ε. At first, we can write that A_ε is a relatively compact perturbation of the operator $(1 + |\partial_x \eta_\varepsilon|^2)^{-\frac{3}{2}} B_\varepsilon$ where

$$B_\varepsilon \varphi = -\beta\, \partial_x^2 \varphi + \alpha\varphi - M^{-1}\varphi$$

and thus, it suffices to study the essential spectrum of B_ε. The next step is to prove that M^{-1} is a relatively compact perturbation of $M_0^{-1} = -\partial_x G[0]^{-1}\partial_x$ which is also well-defined thanks to Lemma 4.5. As in the end of the proof of Proposition 4.2, it suffices to prove that

$$M = (\mathrm{Id} + \mathcal{C})M_0 \tag{4.8}$$

with $\mathcal{C}$ is compact as an operator in $\mathcal{B}(H^2, H^1)$. Indeed, if this fact is proven, as in the proof of Proposition 4.2, we immediately get from this that

$$M^{-1} = M_0^{-1}\big(\mathrm{Id} - (\mathrm{Id} + \mathcal{C})^{-1}\mathcal{C}\big) = M_0^{-1} + \mathcal{K}$$

where $\mathcal{K}$ is a compact operator in $\mathcal{B}(H^2, L^2)$. Consequently, B_ε is a relatively compact perturbation of

$$A_0 = -\beta\, \partial_x^2 + \alpha\, \mathrm{Id} + M_0^{-1} = \beta|D_x|^2 + \alpha\, \mathrm{Id} - \frac{|D_x|}{\tanh(|D_x|)}$$

and hence we get from Lemma 4.4 that its essential spectrum is contained in $[\varepsilon^2, +\infty)$.

To prove (4.8), we can use (4.5), to get

$$\partial_x^{-1}\big(G[\eta_\varepsilon] - G[0]\big)\partial_x^{-1}\varphi = \int_0^1 \Big(-\partial_x^{-1}G[s\eta_\varepsilon](\eta_\varepsilon Z(s\eta_\varepsilon, \partial_x^{-1}\varphi)) - \big(\eta_\varepsilon(\varphi - sZ(s\eta_\varepsilon, \partial_x^{-1}\varphi)\partial_x\eta_\varepsilon)\big)\Big)ds.$$

Note that this formula is meanigful since $\partial_x^{-1}G[s\eta_\varepsilon]$, $G[s\eta_\varepsilon]\partial_x^{-1}$ and thus $Z(s\eta_\varepsilon, \partial_x^{-1}\cdot)$ are well defined bounded operators on L^2 thanks to (3.3) in Proposition 3.1. By using (3.23) in Proposition 3.5 we can write

$$\partial_x^{-1}G[s\eta_\varepsilon] = \partial_x^{-1}|D_x| + \mathcal{R}_1, \quad G[s\eta_\varepsilon]\partial_x^{-1} = \partial_x^{-1}|D_x| + \mathcal{R}_2$$

where $\mathcal{R}_1$, $\mathcal{R}_2$ are bounded operators from H^s to H^{s+1}. Consequently, as in the proof of Proposition 4.2, we obtain that $\partial_x^{-1}\big(G[\eta_\varepsilon] - G[0]\big)\partial_x^{-1} = M - M_0$ is a compact operator in $\mathcal{B}(H^1, H^{2-\varepsilon})$ and hence in $\mathcal{B}(H^1, H^1)$. Since $M_0^{-1} \in \mathcal{B}(H^2, H^1)$ is invertible we obtain (4.8).

For the study of the eigenvalues, we shall just give a sketch of the argument of Mielke [26] by explaining how the problem can be reduced to the study of the KdV problem. We first get that zero is an eigenvalue from the differentiation of the equation satisfied by the solitary wave. Let us denote by S_ε the scaling map $S_\varepsilon(\eta)(x) = \eta(\varepsilon x)$. Then $S_\varepsilon^{-1}(\eta)(x) = \eta(x/\varepsilon)$. It turns out that in the limit $\varepsilon \to 0$, the operator $\varepsilon^{-2}S_\varepsilon^{-1}A_\varepsilon S_\varepsilon$ is a zero order perturbation of the operator

$$\varepsilon^{-2}S_\varepsilon^{-1}A_0 S_\varepsilon = \beta|D_x|^2 + \varepsilon^{-2}(1+\varepsilon^2) - \varepsilon^{-2}\frac{|\varepsilon D_x|}{\tanh(|\varepsilon D_x|)}\,.$$

For fixed $\xi \in \mathbb{R}$, we have (see Lemma 4.4) the expansion for ε near zero :

$$\beta\xi^2 + \varepsilon^{-2}(1+\varepsilon^2) - \varepsilon^{-2}\frac{|\varepsilon\xi|}{\tanh(|\varepsilon\xi|)} = (\beta - 1/3)\xi^2 + 1 + \mathcal{O}(\varepsilon)$$

which allows to prove that

$$\lim_{\varepsilon\to 0} |\varepsilon^{-2}S_\varepsilon^{-1}A_0 S_\varepsilon(u) - \big(-(\beta - 1/3)\partial_x^2 + 1\big)(u)|_{L^2} = 0, \quad \forall\, u \in H^2(\mathbb{R}). \tag{4.9}$$

Next, we can write

$$\varepsilon^{-2}S_\varepsilon^{-1}A_\varepsilon S_\varepsilon = \varepsilon^{-2}S_\varepsilon^{-1}(A_\varepsilon - A_0)S_\varepsilon + \varepsilon^{-2}S_\varepsilon^{-1}A_0 S_\varepsilon\,.$$

and the main point in the proof is to show that

$$\lim_{\varepsilon\to 0}\Big\|\varepsilon^{-2}S_\varepsilon^{-1}(A_\varepsilon - A_0)S_\varepsilon + 3\cosh^{-2}\Big(\frac{x}{2(\beta-1/3)^{1/2}}\Big)\Big\|_{H^2\to L^2} = 0. \tag{4.10}$$

Indeed despite the rather weak link between the operators, one can then deduce that for small ε, the eigenvalues of $\varepsilon^{-2}S_\varepsilon^{-1}A_\varepsilon S_\varepsilon$ are small perturbations of the ones of

$$-(\beta-1/3)\partial_x^2 + 1 - 3\cosh^{-2}\Big(\frac{x}{2(\beta-1/3)^{1/2}}\Big)$$

which is the operator that arises when linearizing the KdV equation about a solitary wave and whose spectrum is very well-known. We refer to [26] p. 2348 for the details. We shall just give the proof of (4.10) which can be easily obtained from some of our previous arguments. Coming back to the definition of A_ε, (4.10) reduces directly to

$$\lim_{\varepsilon\to 0}\Big\|\varepsilon^{-2}S_\varepsilon^{-1}\Big(-\gamma_\varepsilon M^{-1}(\gamma_\varepsilon\cdot) + \frac{|D_x|}{\tanh(|D_x|)}\Big)S_\varepsilon + 3\cosh^{-2}\Big(\frac{x}{2(\beta-1/3)^{1/2}}\Big)\Big\|_{H^2\to L^2} = 0 \tag{4.11}$$

(the other terms involved in the definition of A_ε tend easily to zero). For that purpose, we write the Taylor expansion

$$G[\eta_\varepsilon](\varphi) = G[0](\varphi) + D_\eta G[0](\varphi)\cdot\eta_\varepsilon + r_\varepsilon(\varphi) = A\varphi + B\varphi,$$

where

$$A = G[0], \quad r_\varepsilon(\varphi) = \int_0^1 (1-t)D_\eta^2 G[t\eta_\varepsilon]\varphi\cdot(\eta_\varepsilon,\eta_\varepsilon)\,dt.$$

To compute $M^{-1} = -\partial_x\big(A+B\big)^{-1}\partial_x$, we use a Neumann series expansion to get

$$\partial_x\big(A+B\big)^{-1}\partial_x = \partial_x A^{-1}\partial_x + \sum_{j=1}^{+\infty}(-1)^j\partial_x\big(A^{-1}B\big)^j A^{-1}\partial_x.$$

Let us observe that we can write

$$\partial_x\big(A^{-1}B\big)^j A^{-1}\partial_x = \partial_x A^{-\frac12}\big(A^{-\frac12}BA^{-\frac12}\big)\cdots\big(A^{-\frac12}BA^{-\frac12}\big)A^{-\frac12}\partial_x$$

where $\big(A^{-\frac12}BA^{-\frac12}\big)$ is repeated j times. Note that $\partial_x A^{-\frac12}$ and $A^{-\frac12}\partial_x$ are bounded Fourier multipliers in $\mathcal{B}(H^s,H^{s-\frac12})$ and that by setting $B = B_1 + r_\varepsilon$, we first have

$$A^{-\frac12}B_1A^{-\frac12}\varphi = -G[0]^{\frac12}\big(\eta_\varepsilon G[0]^{\frac12}\varphi\big) - \partial_x G[0]^{-\frac12}\big(\eta_\varepsilon\partial_x G[0]^{-\frac12}\varphi\big)$$

thanks to Lemma 1.1. By using classical commutator estimates, we have

$$\big|A^{-\frac12}B_1A^{-\frac12}\big|_{\mathcal{B}(H^s,H^s)} \le C_s\varepsilon^2.$$

Using estimates in the spirit of (3.3) for the Frechet derivative of $G[\eta]$, we also have

$$\big|A^{-\frac12}r_\varepsilon A^{-\frac12}\big|_{\mathcal{B}(H^s,H^s)} \le C_s\varepsilon^4.$$

Therefore the Neumann series is well-defined and converges and for $\varepsilon \ll 1$ we have the expansion

$$-M^{-1}(\varphi) = -\frac{|D_x|}{\tanh(|D_x|)}(\varphi) - \partial_x G[0]^{-1}D_\eta G[0](G[0]^{-1}\partial_x\varphi)\cdot\eta_\varepsilon + R_\varepsilon(\varphi),$$

with $\|S_\varepsilon^{-1}R_\varepsilon S_\varepsilon\|_{H^2\to L^2} = \mathcal{O}(\varepsilon^3)$. Next, using Lemma 1.1, we may write

$$\partial_x G[0]^{-1}D_\eta G[0](G[0]^{-1}\partial_x\varphi)\cdot\eta_\varepsilon = -\partial_x(\eta_\varepsilon\partial_x\varphi) - \frac{|D_x|}{\tanh(|D_x|)}\Big(\eta_\varepsilon\frac{|D_x|}{\tanh(|D_x|)}(\varphi)\Big). \tag{4.12}$$

After the conjugation with $\varepsilon^{-1}S_\varepsilon$ the operator defining the first term in the right hand-side of (4.12) tends to zero as $\varepsilon \to 0$ in $\mathcal{B}(H^2, L^2)$.

Thanks to [6], the solitary wave $(\eta_\varepsilon, \varphi_\varepsilon)$ may be written as

$$\begin{aligned}\eta_\varepsilon(x) &= -\varepsilon^2 \cosh^{-2}\Big(\frac{\varepsilon x}{2(\beta-1/3)^{1/2}}\Big) + \mathcal{O}(\varepsilon^4),\\ \varphi_\varepsilon(x) &= -2(\beta-1/3)^{1/2}\varepsilon \tanh\Big(\frac{\varepsilon x}{2(\beta-1/3)^{1/2}}\Big) + \mathcal{O}(\varepsilon^3)\,.\end{aligned}$$

Hence the second term in the right hand-side of (4.12), conjugated by $\varepsilon^{-1}S_\varepsilon$ has the same limit as

$$\cosh^{-2}\Big(\frac{x}{2(\beta-1/3)^{1/2}}\Big)\Big(\frac{|\varepsilon D_x|}{\tanh(|\varepsilon D_x|)}\Big)^2,$$

the commutator tending to zero in $\mathcal{B}(H^2, L^2)$. Next, we have that

$$\lim_{\varepsilon\to 0}\Big\|\cosh^{-2}\Big(\frac{x}{2(\beta-1/3)^{1/2}}\Big)\Big(\frac{|\varepsilon D_x|}{\tanh(|\varepsilon D_x|)}\Big)^2 - \cosh^{-2}\Big(\frac{x}{2(\beta-1/3)^{1/2}}\Big)\Big\|_{H^2\to L^2} = 0.$$

Therefore, we obtain that

$$\lim_{\varepsilon\to 0}\Big\|\varepsilon^{-2}S_\varepsilon^{-1}\Big(-\gamma_\varepsilon M^{-1}(\gamma_\varepsilon\cdot) + \gamma_\varepsilon\frac{|D_x|(\gamma_\varepsilon\cdot)}{\tanh(|D_x|)}\Big)S_\varepsilon + \cosh^{-2}\Big(\frac{x}{2(\beta-1/3)^{1/2}}\Big)\Big\|_{H^2\to L^2} = 0$$

and thus (4.11) would be a consequence of

$$\lim_{\varepsilon\to 0}\Big\|\varepsilon^{-2}S_\varepsilon^{-1}\Big(-\gamma_\varepsilon\frac{|D_x|(\gamma_\varepsilon\cdot)}{\tanh(|D_x|)} + \frac{|D_x|}{\tanh(|D_x|)}\Big)S_\varepsilon + 2\cosh^{-2}\Big(\frac{x}{2(\beta-1/3)^{1/2}}\Big)\Big\|_{H^2\to L^2} = 0. \tag{4.13}$$

Coming back to the definition of γ_ε, we obtain that

$$\gamma_\varepsilon = 1 + \varepsilon^2\cosh^{-2}\Big(\frac{\varepsilon x}{2(\beta-1/3)^{1/2}}\Big) + \mathcal{O}(\varepsilon^4), \tag{4.14}$$

in $W^{s,\infty}(\mathbb{R})$, $s \geq 0$. Now, (4.13) follows from (4.14). □

Remark 4.7 (Fixing the value of ε)**.** *From now on, ε will be fixed in the range of validity of Proposition 4.6, Proposition 4.2 and Theorem 1.2. More precisely, from now on we fix an ε such that $\varepsilon \in (0, \varepsilon_0]$, where ε_0 is determined by Proposition 4.6, Proposition 4.2 and Theorem 1.2.*

Let us define η_ε^- and η_ε^0 as the L^2 normalized eigenvalues of A_ε, i.e. such that

$$A_\varepsilon\eta_\varepsilon^- = \lambda_\varepsilon^-\eta_\varepsilon^-, \quad A_\varepsilon\eta_\varepsilon^0 = 0. \tag{4.15}$$

Since A^ε is a self adjoint operator, we get from Proposition 4.6 that there exists $c_\varepsilon > 0$ such that

$$\big(A_\varepsilon\eta, \eta\big) \geq c_\varepsilon|\eta|_{H^1}^2, \quad \forall\,\eta \in H^1(\mathbb{R}), \quad (\eta, \eta_\varepsilon^-) = 0,\ (\eta, \eta_\varepsilon^0) = 0. \tag{4.16}$$

Indeed, Proposition 4.6 yields the weaker bound

$$\big(A_\varepsilon\eta, \eta\big) \geq c_\varepsilon|\eta|_{L^2}^2, \quad \forall\,\eta \in H^1(\mathbb{R}), \quad (\eta, \eta_\varepsilon^-) = 0,\ (\eta, \eta_\varepsilon^0) = 0. \tag{4.17}$$

But using that

$$|(M^{-1}\eta, \eta)| \leq |\eta|_{H^{\frac12}}|M^{-1}\eta|_{H^{-\frac12}} \leq C|\eta|_{H^{\frac12}}^2 \leq C|\eta|_{L^2}|\eta|_{H^1}\,,$$

we obtain that

$$\big(A_\varepsilon\eta, \eta\big) \geq \tilde{c}_\varepsilon\Big(|\eta|_{H^1}^2 - C|\eta|_{L^2}^2\Big), \quad \forall\,\eta \in H^1(\mathbb{R}), \quad (\eta, \eta_\varepsilon^-) = 0,\ (\eta, \eta_\varepsilon^0) = 0. \tag{4.18}$$

A combination of (4.17) and (4.18) gives (4.16). Thanks to Proposition 4.6, we get the following crucial property of $L(k)$.

Proposition 4.8. *The operator $L(0)$ has a unique simple negative eigenvalue and for every k, $L(k)$ has at most one simple negative eigenvalue. Moreover, for every $\varepsilon \in (0, \varepsilon_0)$, there exists $c > 0$ such that for every k, we have that*

$$\big(L(k)U, U\big) \geq c\Big(|U_1|_{H^1}^2 + \Big|\frac{|D_x|}{1+|D_x|^{\frac{1}{2}}}U_2\Big|_{L^2}^2 + \frac{|k|^2}{1+|k|}|U_2|_{L^2}^2\Big), \tag{4.19}$$

for every U such that

$$(U_1, \eta_\varepsilon^-) = 0, \; (U_1, \eta_\varepsilon^0) = 0. \tag{4.20}$$

Note that the estimate (4.19) is uniform for $k \in [0, +\infty[$. In some part of the paper, we shall only need a weaker estimate uniform for $k \in [0, K]$ for some $K > 0$ fixed. In this case, we can deduce from (4.19) that for every U which satisfies (4.20) we have

$$\big(L(k)U, U\big) \geq c_1\Big(|U_1|_{H^1}^2 + \Big|\frac{|D_x|}{1+|D_x|^{\frac{1}{2}}}U_2\Big|_{L^2}^2 + |k|^2|U_2|_{L^2}^2\Big), \quad \forall k, \; |k| \leq K, \tag{4.21}$$

where $c_1 > 0$ depends on K.

Proof of Proposition 4.8. We shall first prove the estimate (4.19). At first, we notice that it suffices to prove that the weaker estimate

$$\big(L(k)U, U\big) \geq c\Big(|U_1|_{H^1}^2 + \Big|\frac{|D_x|}{1+|D_x|^{\frac{1}{2}}}U_2\Big|_{L^2}^2\Big), \quad \forall\, U, \; (U_1, \eta_\varepsilon^-) = 0, \; (U_1, \eta_\varepsilon^0) = 0 \tag{4.22}$$

holds and to combine it with a crude estimate to get the result. Indeed, let us use that

$$\big(L(k)U, U\big) \geq m\big(G_{\varepsilon,k}U_2, U_2\big) - M|U_1|_{H^1}^2 - 2\big|\big(\partial_x\big((v_\varepsilon - 1)U_1\big), U_2\big)\big|\Big) \tag{4.23}$$

where $m > 0$ and $M > 0$ are harmless numbers independent of k which will change from line to line. To estimate the last term, we use that

$$\begin{aligned}
\big|\big(\partial_x\big((v_\varepsilon - 1)U_1\big), U_2\big)\big| &\leq |\partial_x U_2|_{H^{-\frac{1}{2}}}\, |(v_\varepsilon - 1)U_1|_{H^{\frac{1}{2}}} \leq M\Big|\frac{|D_x|}{1+|D_x|^{\frac{1}{2}}}U_2\Big|_{L^2} |(v_\varepsilon - 1)U_1|_{H^1} \\
&\leq M\Big|\frac{|D_x|}{1+|D_x|^{\frac{1}{2}}}U_2\Big|_{L^2}^2 + |v_\varepsilon - 1|_{W^{1,\infty}}^2 |U_1|_{H^1}^2 \\
&\leq M\Big(\Big|\frac{|D_x|}{1+|D_x|^{\frac{1}{2}}}U_2\Big|_{L^2}^2 + |U_1|_{H^1}^2\Big).
\end{aligned}$$

Consequently, we get from (4.23) that

$$\big(L(k)U, U\big) \geq m\big(G_{\varepsilon,k}U_2, U_2\big) - M\Big(|U_1|_{H^1}^2 + \Big|\frac{|D_x|}{1+|D_x|^{\frac{1}{2}}}U_2\Big|_{L^2}^2\Big).$$

Next, we can add (4.22) times a large constant and the last estimate to get

$$\big(L(k)U, U\big) \geq m\Big(|U_1|_{H^1}^2 + \Big|\frac{|D_x|}{1+|D_x|^{\frac{1}{2}}}U_2\Big|_{L^2}^2 + \big(G_{\varepsilon,k}U_2, U_2\big)\Big).$$

Finally, we can use (3.4), which gives in particular that

$$\big(G_{\varepsilon,k}U_2, U_2\big) \geq \frac{|k|^2}{1+|k|}|U_2|_{L^2}^2$$

to get (4.19), assuming (4.22).

It remains to prove that (4.22) holds. Thanks to the monotonicity of $G_{\varepsilon,k}$ with respect to k proven in Proposition 3.1 ii) and since we also obviously have

$$\big(-P_{\varepsilon,k_1}U_1,U_1\big)\geq\big(-P_{\varepsilon,k_2}U_1,U_1\big),\quad |k_1|\geq|k_2|,$$

we get in particular that

$$\big(L(k)U,U\big)\geq(L(0)U,U),\quad \forall\, U\in H^1\times H^{\frac{1}{2}}. \tag{4.24}$$

Next, thanks to (4.6), we have

$$(L(0)U,U)=(A_\varepsilon U_1,U_1)+\Big(M\big(\partial_x U_2-M^{-1}(\gamma_\varepsilon U_1)\big),\partial_x U_2-M^{-1}(\gamma_\varepsilon U_1)\Big). \tag{4.25}$$

Consequently, we can use the assumption that

$$(U_1,\eta_\varepsilon^-)=0,\ (U_1,\eta_\varepsilon^0)=0$$

and hence (4.16) and the coercivity of Q in Lemma 4.5 to get

$$\big(L(k)U,U\big)\geq c\Big(|U_1|_{H^1}^2+\Big|\partial_x U_2-M^{-1}(\gamma_\varepsilon U_1)\Big|_{H^{-\frac{1}{2}}}^2\Big)$$

for some $c>0$ (c actually depends on ε but we do not care on this dependence here and thus we omit it in our notations). The expansion of the second term and the Cauchy-Schwarz inequality give

$$\Big|\partial_x U_2-M^{-1}(\gamma_\varepsilon U_1)\Big|_{H^{-\frac{1}{2}}}^2\geq|\partial_x U_2|_{H^{-\frac{1}{2}}}^2+\big|M^{-1}(\gamma_\varepsilon U_1)\big|_{H^{-\frac{1}{2}}}^2-2|\partial_x U_2|_{H^{-\frac{1}{2}}}\big|M^{-1}(\gamma_\varepsilon U_1)\big|_{H^{-\frac{1}{2}}}.$$

Consequently, by using the inequality

$$2ab\leq\delta a^2+\frac{1}{\delta b^2},\quad \forall\, a,\,b\geq 0,\,\forall\,\delta>0, \tag{4.26}$$

we get

$$\Big|\partial_x U_2-M^{-1}(\gamma_\varepsilon U_1)\Big|_{H^{-\frac{1}{2}}}^2\geq(1-\delta)|\partial_x U_2|_{H^{-\frac{1}{2}}}^2-\big(\frac{1}{\delta}-1\big)\big|M^{-1}(\gamma_\varepsilon U_1)\big|_{H^{-\frac{1}{2}}}^2$$

where $\delta<1$ will be chosen carefully later. Next, since $M^{-1}\in\mathcal{B}(H^{\frac{1}{2}},H^{-\frac{1}{2}})$ and $\gamma_\varepsilon\in W^{1,\infty}$, we can use the crude estimate

$$\big|M^{-1}(\gamma_\varepsilon U_1)\big|_{H^{-\frac{1}{2}}}\leq C|\gamma_\varepsilon U_1|_{H^{\frac{1}{2}}}\leq C|\gamma_\varepsilon U_1|_{H^1}\leq C|U_1|_{H^1}$$

for some $C>0$. This yields

$$\Big|\partial_x U_2-M^{-1}(\gamma_\varepsilon U_1)\Big|_{H^{-\frac{1}{2}}}^2\geq(1-\delta)|\partial_x U_2|_{H^{-\frac{1}{2}}}^2-C\big(\frac{1}{\delta}-1\big)|U_1|_{H^1}^2$$

and hence we find that

$$\big(L(k)U,U\big)\geq c\Big(\big(1-C\big(\frac{1}{\delta}-1\big)\big)|U_1|_{H^1}^2+(1-\delta)|\partial_x U_2|_{H^{-\frac{1}{2}}}^2\Big).$$

To conclude, it suffices to choose δ in the non-empty interval $\big(\frac{C}{1+C},1\big)$ thus for example

$$\delta=\frac{1}{2}\Big(1+\frac{C}{1+C}\Big).$$

This proves (4.22) which in turn implies (4.19).

Next, we shall prove that for every k, $L(k)$ has at most one simple negative eigenvalue. Thanks to Proposition 4.6, we notice that the quadratic form $(\cdot,A_\varepsilon\cdot)$ is nonnegative on $(\eta_\varepsilon^-)^\perp$. Thanks to (4.24), (4.25) and Lemma 4.5, this yields that the quadratic form $(L(k)\cdot,\cdot)$ is nonnegative on $(\eta_\varepsilon^-,0)^\perp$. By contradiction, we obtain that $L(k)$ has at most one simple negative eigenvalue. Indeed, otherwise $L(k)$ would have an invariant subspace at least two-dimensional on which the quadratic

form $(L(k)\cdot,\cdot)$ is strictly negative. Since this subspace must meet $(\eta_\varepsilon^-,0)^\perp$ in a non-trivial way, this yields a contradiction.

Now, we shall prove that a negative eigenvalue of $L(0)$ indeed exists. Since $L(0)$ is self-adjoint with essential spectrum included in $[0,+\infty[$ thanks to Proposition 4.2, it suffices to prove that there exists U such that $(L(0)U,U)<0$. In view of (4.25), a good candidate would be $U=(U_1,U_2)$ such that

$$U_1=\eta_\varepsilon^-, \quad \partial_x U_2 = M^{-1}(\gamma_\varepsilon U_1).$$

Nevertheless, the second equation above does not necessarily have a solution U_2 in L^2. To fix this difficulty we shall define a sequence $U^n=(U_1^n,U_2^n)$ in L^2 and prove that $(L(0)U^n,U^n)$ becomes negative for n sufficiently large. We set

$$U_1^n=\eta_\varepsilon^-, \quad U_2^n=\chi(nD_x)\partial_x^{-1}M^{-1}(\gamma_\varepsilon\eta_\varepsilon^-)$$

where χ is a smooth bounded function such that $\chi(\xi)=0$ on $(-1/2,1/2)$ and $\chi(\xi)=1$ for $|\xi|\geq 1$. Next, thanks to Lemma 4.5, we notice that

$$\begin{aligned}
&\Big|\Big(M\big(\partial_x U_2^n - M^{-1}(\gamma_\varepsilon U_1^n)\big), \partial_x U_2^n - M^{-1}(\gamma_\varepsilon U_1^n)\Big)\Big| \\
&\leq C \quad \big|\partial_x U_2^n - M^{-1}(\gamma_\varepsilon U_1^n)\big|_{H^{-\frac12}}^2 \\
&\leq C \quad \big|\big(\chi(nD_x)-1\big)M^{-1}(\gamma_\varepsilon\eta_\varepsilon^-)\big|_{H^{-\frac12}}^2
\end{aligned}$$

and hence, since $M^{-1}(\gamma_\varepsilon\eta_\varepsilon^-)\in H^{-\frac12}$, we get from the dominated convergence theorem that

$$\lim_{n\to\infty}\Big(M\big(\partial_x U_2^n - M^{-1}(\gamma_\varepsilon U_1^n)\big), \partial_x U_2^n - M^{-1}(\gamma_\varepsilon U_1^n)\Big)=0.$$

Consequently, thanks to (4.25), we get that $(L(0)U^n,U^n)$ is negative for n sufficiently large. This ends the proof of Proposition 4.8. $\square$

5. Study of the linearized about the solitary wave operator $JL(k)$. Transverse linear instability.

As in [32], we shall say that the linearized equation (2.1) or equivalently (2.3) has an unstable eigenmode with transverse frequency k and amplification parameter σ with $\mathrm{Re}\,\sigma>0$ if there is a non-trivial solution of

$$\partial_t V = JLV \tag{5.1}$$

under the form

$$V(t,x,y)=e^{\sigma t}e^{iky}U(x) \tag{5.2}$$

with $U\in H^2\times H^1$. This provides a non-trivial solution of (2.1) via (2.7). By substitution of the ansatz (5.2) in the equation (5.1), we get the resolvent equation

$$\sigma U = JL(k)U. \tag{5.3}$$

Note that if there is a solution $U\in H^{k+1}\times H^k$, we find from the first equation of the system that $G_{\varepsilon,k}U_2\in H^k$ thus since $G_{\varepsilon,k}$ is a first order elliptic operator (see Corollary 3.6), we get that $U_2\in H^{k+1}$. Next, the second equation gives that $P_{\varepsilon,k}U_1\in H^k$ and hence since $P_{\varepsilon,k}$ is a second order elliptic operator, we get that $U_1\in H^{k+2}$. Consequently, one can get by induction that an unstable eigenmode U is necessarily smooth, $U\in H^\infty\times H^\infty$.

5.1. **Location of unstable eigenmodes.** We start with a Lemma which gives a crucial preliminary information on the possible solutions of (5.3).

Lemma 5.1. *For every $\varepsilon \in (0, \varepsilon_0)$, $\sigma(JL(0)) \subset i\mathbb{R}$. Moreover, for every $k \neq 0$, $JL(k)$ has at most one unstable eigenmode which is necessarily simple. Finally, if (5.3) has an unstable mode with amplification parameter σ then $\sigma \in \mathbb{R}$.*

Proof. The first part is a direct consequence of the one-dimensional stability result of Mielke [26]. For the second part, we follow Pego-Weinstein [29]. Suppose that there exist linearly independent u_1 and u_2 such that $JL(k)u_j = \sigma_j u_j$, $\mathrm{Re}(\sigma_j) > 0$, $j = 1, 2$. Set $v_j(t) \equiv e^{\sigma_j t} u_j$. Thus $\partial_t v_j = JL(k)v_j$. Next we observe that thanks to the symmetry of $L(k)$ and the skew-symmetry of J, we have $\partial_t(L(k)v_1(t), v_2(t)) = 0$. This implies that for every real t, $e^{t(\sigma_1+\overline{\sigma_2})}(L(k)u_1, u_2) = (L(k)u_1, u_2)$. Using that $\mathrm{Re}(\sigma_j) > 0$, we obtain that $(L(k)u_1, u_2) = 0$. Since we know that $L(k)$ has at most one negative direction, we obtain that there exists a complex number γ such that $(L(k)(u_1 + \gamma u_2), u_1 + \gamma u_2) \geq 0$. Therefore $(L(k)u_1, u_1) + |\gamma|^2(L(k)u_2, u_2) \geq 0$. By taking the scalar product of $JL(k)u_j = \sigma_j u_j$ by $L(k)u_j$ and taking the real part, we obtain that $(L(k)u_j, u_j) = 0$. Therefore $u_1 + \gamma u_2$ is in the kernel of $L(k)$. This in turn implies that $\sigma_1 u_1 + \gamma\sigma_2 u_2 = 0$ which is a contradiction. Next we can show similarly that an unstable eigenvalue can not be of multiplicity higher than 1. Indeed, if we suppose that u_1 and u_2 are such that $JL(k)u_1 = \sigma u_1$ and $JL(k)u_2 = \sigma u_2 + u_1$ then we may consider v_1 and v_2 defined as $v_1(t) = e^{\sigma t} u_1$, $v_2(t) = e^{\sigma t}(u_2 + t u_1)$ and obtain a contradiction as above. Finally, we observe that if (5.3) has a nontrivial solution for some σ and k, then by taking the complex conjugate, we obtain that (5.3) has a nontrivial solution with the same k and σ replaced by $\bar{\sigma}$. If σ is not real this contradicts the previous analysis which showed that for each k there is at most one σ such that (5.3) has a nontrivial solution. This completes the proof of Lemma 5.1. $\square$

In the next lemma, we give a further localization where unstable eigenmodes must be sought. We have the following statement giving further information on the location of the possible unstable eigenmodes.

Proposition 5.2 (Location of unstable eigenmodes)**.** *We have the following information on the location of unstable eigenmodes:*

i) *There exists $K > 0$ such that if $|k| > K$ then there is no unstable eigenmode with transverse frequency k and amplification parameter σ satisfying $Re(\sigma) > 0$.*
ii) *There exists $M > 0$ such that for every k , $|k| \leq K$, there is no unstable eigenmode with transverse frequency k and with amplification parameter σ satisfying $Re(\sigma) \geq M$.*

Proof of Proposition 5.2. By taking the scalar product of (5.3) by $L(k)U$ and then taking the real part, we get that

$$\mathrm{Re}(\sigma)(L(k)U, U) = 0$$

and hence if Re $\sigma > 0$, this yields

$$(L(k)U, U) = 0. \tag{5.4}$$

Next, we get by a very crude estimate that

$$(L(k)U, U) \geq c|U_1|_{H^1}^2 + k^2|U_1|_{L^2}^2 + (G_{\varepsilon,k}U_2, U_2) - C\big(|U_1|_{H^1}\,|U_2|_{L^2} + |U_1|_{L^2}^2\big)$$

where $c > 0$, $C > 0$ are independent of k. Thanks to (3.4), for k large (actually $k \geq 1$ is sufficient), we have

$$(G_{\varepsilon,k}u, u) \geq c\Big(\Big|\frac{|D_x|}{1 + |D_x|^{\frac{1}{2}}}u\Big|_{L^2}^2 + |k||u|_{L^2}^2\Big).$$

Consequently, we obtain

$$(5.5)\quad (L(k)U,U) \geq c\Big(|U_1|_{H^1}^2 + k^2|U_1|_{L^2}^2 + \Big|\frac{|D_x|}{1+|D_x|^{\frac12}}u\Big|_{L^2}^2 + |k||U_2|_{L^2}^2\Big) - C\big(|U_1|_{H^1}\,|U_2|_{L^2} + |U_1|_{L^2}^2\big)$$

and hence, thanks to a new use of (4.26), we easily get that for k sufficiently large

$$(L(k)U,U) \geq c\Big(|U_1|_{H^1}^2 + |U_2|_{H^{\frac12}}^2\Big). \tag{5.6}$$

In particular, we get from (5.4) that $U=0$. This proves i).

We turn to the proof of ii). We use a decomposition of $L(k)$ under the form

$$L(k) = L_0(k) + L_1, \tag{5.7}$$

where

$$L_0(k) = \begin{pmatrix} -P_{\varepsilon,k}+\alpha & 0 \\ 0 & G_{\varepsilon,k} \end{pmatrix}, \quad L_1 = \begin{pmatrix} (v_\varepsilon-1)\partial_x Z_\varepsilon & (v_\varepsilon-1)\partial_x \\ -\partial_x((v_\varepsilon-1)\cdot) & 0 \end{pmatrix}. \tag{5.8}$$

Note that L_0 is a real-symmetric operator. By taking the scalar product of (5.3) with $L_0(k)U$, we find

$$\mathrm{Re}(\sigma)(L_0(k)U,U) = \mathrm{Re}\,\big(JL_1U, L_0(k)U\big). \tag{5.9}$$

By using an integration by parts and (3.4), we get that

$$(L_0(k)U,U) \geq c\Big(|U_1|_{H^1}^2 + \Big|\frac{|D_x|}{1+|D_x|^{\frac12}}U_2\Big|_{L^2}^2 + |k|^2\,|U_2|_{L^2}^2\Big) \tag{5.10}$$

for some $c>0$. To estimate the right-hand side of (5.9), we need to estimate the following quantities:

$$\begin{aligned} I &= \big|\mathrm{Re}\,\big((v_\varepsilon-1)(\partial_x Z_\varepsilon)U_1, G_{\varepsilon,k}U_2\big)\big|, \\ II &= \big|\mathrm{Re}\,\big(v_\varepsilon-1)\partial_x U_2, G_{\varepsilon,k}U_2\big)\big|, \\ III &= \big|\mathrm{Re}\,\big(-\partial_x((v_\varepsilon-1)U_1), (P_{\varepsilon,k}+\alpha)U_1\big)\big|. \end{aligned}$$

The term I is easy to bound, it suffices to use (3.3) to get

$$I \leq C|U_1|_{H^1}\Big(\Big|\frac{|D_x|}{1+|D_x|^{\frac12}}U_2\Big|_{L^2} + |k|\,|U_2|_{L^2}\Big).$$

The estimate of II follows from the commutator estimate of Proposition 3.8 which yields

$$II \leq C\Big(\Big|\frac{|D_x|}{1+|D_x|^{\frac12}}U_2\Big|_{L^2}^2 + |k|^2\,|U_2|_{L^2}^2\Big).$$

Finally, by using integration by parts, we also easily get that

$$III \leq C|U_1|_{H^1}^2.$$

We have thus proven that

$$\big|\big(JL_1U, L_0(k)U\big)\big| \leq C\Big(|U_1|_{H^1}^2 + \Big|\frac{|D_x|}{1+|D_x|^{\frac12}}U_2\Big|_{L^2}^2 + |k|^2\,|U_2|_{L^2}^2\Big). \tag{5.11}$$

Consequently, we obtain from (5.9), (5.10) that

$$c\,\mathrm{Re}(\sigma)\Big(|U_1|_{H^1}^2 + \Big|\frac{|D_x|}{1+|D_x|^{\frac12}}U_2\Big|_{L^2}^2 + |k|^2\,|U_2|_{L^2}^2\Big) \leq C\Big(|U_1|_{H^1}^2 + \Big|\frac{|D_x|}{1+|D_x|^{\frac12}}U_2\Big|_{L^2}^2 + |k|^2\,|U_2|_{L^2}^2\Big)$$

for some constant $C>0$, depending on K but independent of σ. This yields that $U=0$ if Re (σ) is sufficiently large. This proves ii). This completes the proof of Proposition 5.2. □

5.2. **Existence of an unstable eigenmode.**

Theorem 5.3 (Linear instability)**.** *There exists $\sigma > 0$ and $k \neq 0$ and a nontrivial $U \in H^2 \times H^1$ such that*

$$JL(k)U = \sigma U.$$

To prove the existence of an unstable eigenmode, we shall follow the general method presented in [32]. Note that this result was proven in [18] by using a different formulation of the water waves equations.

Proof of Theorem 5.3. Let us set $M(k) \equiv JL(k)J$. Note that

$$(M(k)u, u) = -(L(k)Ju, Ju), \quad \forall\, u \in H^1(\mathbb{R}) \times H^{\frac{1}{2}}(\mathbb{R}). \tag{5.12}$$

Moreover, since J is invertible matrix, we get from Proposition 4.8 that there exists $\lambda > 0$ and $v, w \in H^1 \times H^{\frac{1}{2}}$ such that $M(0)v = \lambda v$, $M(0)w = 0$ and that for some $c > 0$

$$(M(0)z, z) \leq -c\Big(|z_1|^2_{H^1} + \Big|\frac{|D_x|}{1 + |D_x|^{\frac{1}{2}}} z_2\Big|^2_{L^2}\Big), \quad \forall\, z = (z_1, z_2), \quad (z, v) = 0,\ (z, w) = 0. \tag{5.13}$$

Next we set

$$f(k) \equiv \sup_{z \in H^1 \times H^{1/2}, |z|_{L^2 \times L^2} = 1} (M(k)z, z)\,.$$

Since $(M(0)v, v) > 0$, we already know that $f(0) > 0$. Moreover, from Proposition 3.1 ii) and the obvious monotonicity of $P_{\varepsilon,k}$, we get that $M(k)$ is strictly decreasing in k on $\mathbb{R}_+$ as a symmetric operator. In particular, this yields that $f(k)$ is decreasing on $\mathbb{R}_+$. By using (5.12) and (5.6), we obtain that for $k \gg 1$ one has $f(k) < 0$. Therefore, since f is continuous, there exists a smallest $k_0 > 0$ such that $f(k_0) = 0$. The following lemma is the key element in the proof of the existence of an unstable mode.

Lemma 5.4. *There exists $k_1 \in (0, k_0]$ such that* $\dim(\mathrm{Ker}(M(k_1)) = 1$. *Moreover, for every $F \in L^2 \times L^2$ orthogonal to* $\mathrm{Ker}(M(k_1))$ *there exists a unique $v \in H^2 \times H^1$ orthogonal to* $\mathrm{Ker}(M(k_1))$ *such that $M(k_1)v = F$.*

Proof. Thanks to Proposition 4.2, we already know that the essential spectrum of $M(k)$ is in $\mathbb{R}_-$ and for $k \neq 0$ it is even included in $(-\infty, -m)$ for some $m > 0$. Indeed, we have

$$M(k) = -JL(k)J^{-1}.$$

Thanks to the classical variational characterization of the largest eigenvalue, and since f is strictly decreasing, we get that for $k \in (0, k_0)$, the largest eigenvalue of $M(k)$ is positive. Moreover, thanks to Proposition 4.8, we further obtain that there is exactly one positive eigenvalue of $M(k)$ for $k \in (0, k_0)$. Indeed, if there were at least two positive eigenvalues of $M(k)$ for some positive k, we would get by the monotonicity of $M(k)$ that $M(0)$ is positive on a subspace of dimension at least two and this contradicts the fact that $L(0)$ has a unique simple negative eigenvalue.

If for some $k \in (0, k_0)$ the kernel of $M(k)$ is non-trivial then, the value k_1 that we are looking for is this k. If for every $k \in (0, k_0)$, the kernel of $M(k)$ is trivial then we have $k_1 = k_0$. Indeed, by definition of k_0, the kernel of $M(k_0)$ is non-trivial and by the classical variational characterization of eigenvalues, 0 is the largest eigenvalue of $M(k_0)$. Moreover, this eigenvalue is simple. Indeed, for $0 < k < k_0$, we have that $M(k)$ is strictly negative on a subspace of codimension 1 (this is a consequence of the fact that the kernel of $M(k)$ is assumed to be trivial for $k < k_0$ and of the fact that $M(k)$ has a unique positive eigenvalue for k, $0 < k < k_0$). Consequently, by monotonicity, we get that $M(k_0)$ is also strictly negative on a subspace of codimension 1. Consequently, it cannot vanish on a subspace of dimension at least two.

The second part in the statement of the lemma is a consequence of the fact that $M(k_1)$ is symmetric and Fredholm index zero. Indeed, 0 is not in the essential spectrum of $M(k_1)$ thanks to Theorem 4.2 since $k_1 \neq 0$. This completes the proof of Lemma 5.4. $\square$

We next finish the proof of Theorem 5.3, i.e. we prove the existence of an unstable mode. Since J is invertible, it is equivalent to prove that there exist $k \neq 0$, $\sigma > 0$ and $u \in H^2 \times H^1$ different from zero such that

$$M(k)u = \sigma Ju\,.$$

Let k_1 be the number defined in Lemma 5.4 with corresponding kernel spanned by u. We need to solve $F(v, k, \sigma) = 0$, with $\sigma > 0$, where $F(v, k, \sigma) \equiv M(k)v - \sigma Jv$. We have that $F(u, k_1, 0) = 0$. We look for v as $v = u + w$, where

$$w \in u^{\perp} \equiv \{v \in H^2 \times H^1 \,:\, (v, u) = 0\}.$$

Therefore we need to solve $G(w, k, \sigma) = 0$ with $\sigma > 0$, where

$$G(w, k, \sigma) = M(k)u + M(k)w - \sigma Ju - \sigma Jw, \quad w \in u^{\perp}\,.$$

We have that

$$D_{v,k}G(0, k_1, 0)[w, \mu] = \mu\Big[\frac{d}{dk}M(k)\Big]_{k=k_1} u + M(k_1)w\,.$$

By, using Lemma 5.4, we shall obtain that $D_{v,k}G(0, k_1, 0)$ is a bijection form $u^{\perp} \times \mathbb{R}$ to $L^2 \times L^2$ if we establish that

$$\Big(\Big[\frac{d}{dk}M(k)\Big]_{k=k_1} u, u\Big) < 0\,.$$

By explicit computation, we have

$$\Big[\frac{d}{dk}M(k)\Big]_{k=k_1} = J\begin{pmatrix} 2k_1\big(1 + (\partial_x\eta_\varepsilon)^2\big)^{-\frac{1}{2}} & 0 \\ 0 & \Big[\frac{d}{dk}G_{\varepsilon,k}\Big]_{k=k_1} \end{pmatrix} J.$$

From Proposition 3.1 ii), we have that $\Big[\frac{d}{dk}G_{\varepsilon,k}\Big]_{k=k_1}$ is a nonnegative operator. Therefore, we obtain

$$\Big(\Big[\frac{d}{dk}M(k)\Big]_{k=k_1} u, u\Big) \leq -2k_1 \int_{\mathbb{R}} \frac{|u_2|^2}{\big(1 + (\partial_x\eta_\varepsilon)^2\big)^{\frac{1}{2}}}\, dx < 0$$

Indeed, we have from the structure of $L(k)$ that u_2 does not vanish identically: assume that u_2 vanishes identically, then $M(k_1)u = 0$ gives that $G_{\varepsilon,k_1}u_1 = 0$ and hence from Proposition 3.1 iii), we get $u_1 = 0$ which is impossible.

Consequently, we have shown that $D_{v,k}G(0, k_1, 0)$ is a bijection form $u^{\perp} \times \mathbb{R}$ to $L^2 \times L^2$ and we can apply the implicit function theorem, in order to complete the proof of Theorem 5.3. $\square$

5.3. **Essential spectrum of $JL(k)$.**

Proposition 5.5. *For $\varepsilon \in (0, \varepsilon_0)$, the essential spectrum of $JL(k)$ is included in $i\mathbb{R}$, for every k.*

Proof. We can use the most restrictive definition for the essential spectrum i.e. following [20], we say that λ is not in the essential spectrum if λ is an isolated eigenvalue of finite multiplicity. Note that since we already know by Lemma 5.1 that $JL(k)$ has at most one unstable eigenvalue, it still suffices following [20] to prove that $\lambda - JL(k)$ is Fredholm with zero index for $\mathrm{Re}(\lambda) \neq 0$. Moreover, the case $k = 0$ is already given by Lemma 5.1, consequently, it suffices to consider the case $k \neq 0$ only. We shall proceed in a similar way as in the proof of Proposition 4.2. Since we are in the case $k \neq 0$, we can use the decomposition (4.4) for $\gamma = 0$. This yields

$$L(k) = A_1 A_2(k) L_4(k) B_2(k) B_1 + \mathcal{K}$$

where, using the notation

$$A_2(k) = A_2(0,k), \quad B_2(k) = B_2(0,k), \quad L_4(k) = L_4(0,k)$$

we have that A_1, A_2, B_1, B_2 are bounded invertible operators and that $\mathcal{K}$ is a relatively compact perturbation. We thus have

$$\lambda - JL(k) = \lambda - JA_1 A_2(k) L_4(k) B_2(k) B_1 + \tilde{K}$$

where $\tilde{K}$ is a relatively compact perturbation. Next, we can write that

$$\begin{aligned}\lambda - JL(k) = \quad & JA_1 J^{-1}\big(\lambda - JA_2 L_4 B_2\big) B_1 \\ & + \lambda J A_1 \big(A_1^{-1} - \mathrm{Id}\big) J^{-1} + \lambda J A_1 J^{-1}\big(\mathrm{Id} - B_1\big) + \tilde{K}.\end{aligned}$$

Since the matrices $A_1^{-1} - \mathrm{Id}$ and $\mathrm{Id} - B_1$ have exponentially decreasing coefficients, we find again that

$$\lambda J A_1 \big(A_1^{-1} - \mathrm{Id}\big) J^{-1} + \lambda J A_1 J^{-1}\big(\mathrm{Id} - B_1\big)$$

is a relatively compact perturbation. Consequently, to prove Proposition 5.5, it suffices to prove that $\lambda - JA_2 L_4 B_2$ is invertible for $\mathrm{Re}\,\lambda \neq 0$.

The operator L_4 has been studied in the proof of Proposition 4.2. We have proven that its spectrum is included in $(0, +\infty)$. Since it is moreover a symmetric operator, we obtain that

$$(L_4 U, U) \geq c|U|_{L^2}^2, \quad \forall U \in H^2 \times H^1 \tag{5.14}$$

for some $c > 0$. Since by an integration by parts and (3.4), we have

$$(L_4 U, U) \geq c|\partial_x U_1|_{L^2}^2 + c|U_2|_{H^{\frac12}}^2 - C|U_1|_{L^2}^2, \tag{5.15}$$

we can combine the two estimates (5.14), (5.15) to get

$$(L_4 U, U) \geq c_0\Big(|U_1|_{H^1}^2 + |U_2|_{H^{\frac12}}^2\Big), \quad \forall U \in H^1 \times H^{\frac12} \tag{5.16}$$

for some $c_0 > 0$. Finally, let us notice that A_2 and B_2 are bounded invertible operators on H^s for every s and that $B_2 = A_2^*$. This implies that the operator $L_5(k) = A_2 L_4 B_2$ is a symmetric operator which satisfies thanks to (5.16)

$$(L_5 U, U) \geq c\Big(|U_1|_{H^1}^2 + |U_2|_{H^{\frac12}}^2\Big), \quad \forall U \in H^1 \times H^{\frac12} \tag{5.17}$$

for some $c > 0$. We shall use this property of L_5 to prove that the operator $\lambda - JL_5$ is invertible if $\mathrm{Re}\,\lambda \neq 0$. For $F \in L^2 \times L^2$, we want to prove that the equation

$$\big(\lambda - JL_5(k)\big) U = F \tag{5.18}$$

has a unique solution $U \in H^2 \times H^1$. The estimate (5.17) immediately gives that there is at most one solution. Indeed, if

$$\big(\lambda - JL_5\big) U = 0,$$

by taking the scalar product and the real part with $L_5 U$, we find that

$$\mathrm{Re}\,(\lambda)\,(L_5 U, U) = 0$$

and hence since $\mathrm{Re}\,\lambda \neq 0$, we get from (5.17) that $U = 0$.

To prove that there exists a solution to (5.18), we shall use a classical approach based on a duality argument combined with an a priori bound which is typically used in the context of evolution equations. To solve (5.18) for $F \in L^2 \times L^2$, we look U under the form $U = JV$ and thus we need to solve

$$AV = J^{-1} F, \quad A = \lambda \mathrm{Id} - L_5 J\,.$$

Since $A^* = \bar{\lambda}\mathrm{Id} + JL_5$, we get

$$|A^\star V|_{H^1 \times H^{1/2}} \geq c|V|_{H^1 \times H^{1/2}}\,. \tag{5.19}$$

Indeed, it suffices to consider $(A^\star V, L_5 V)$ and apply (5.17). Next, we define $\mathcal{F}$ as

$$\mathcal{F} = \{U \in H^1 \times H^{1/2} \,:\, \exists\, V \in H^1 \times H^{1/2},\, A^\star(V) = U\}\,.$$

Thanks to (5.19), $\mathcal{F}$ is a closed set of $H^1 \times H^{1/2}$. Indeed, let $U_n \in \mathcal{F}$ that converges to some limit U in $H^1 \times H^{1/2}$. Then there exists $V_n \in H^1 \times H^{1/2}$ such that $U_n = A^\star(V_n)$ and thanks to (5.19) V_n converges to some limit V in $H^1 \times H^{1/2}$. In particular $A^\star(V_n)$ converges in $H^{-2} \times H^{-2}$ to $A^\star V$ which allows to identify U and $A^\star V$, i.e. $U = A^\star V$ and thus get that $U \in \mathcal{F}$. We have thus proven that $\mathcal{F}$ is a closed set of $H^1 \times H^{1/2}$. Now, we define the linear form $l : H^1 \times H^{1/2} \to \mathbb{C}$ as

$$l(U) = \begin{cases} (J^{-1}F, V), & \text{if} \quad U \in \mathcal{F} \quad \text{with} \quad U = A^\star V, \\ 0, \quad \text{if} \quad U \in \mathcal{F}^\perp \end{cases}$$

Using again (5.19) and the fact that $\mathcal{F}$ is closed in $H^1 \times H^{1/2}$, we obtain that l is continuous on $H^1 \times H^{\frac{1}{2}}$ and therefore there exists $V \in H^{-1} \times H^{-1/2}$ such that

$$l(U) = (V, U), \quad \forall\, U \in H^1 \times H^{1/2}\,.$$

If $U = A^\star W$ with $W \in H^3 \times H^2$ then $U \in \mathcal{F}$. Therefore

$$(AV, W) = (V, A^\star W) = (J^{-1}F, W), \quad \forall\, W \in H^3 \times H^2\,.$$

Hence $AV = J^{-1}F$ and thus $U = JV$ is a solution of (5.18). Moreover thanks to the elliptic regularity $U \in H^2 \times H^1$. This ends the proof of Proposition 5.5. $\square$

As a consequence of the Lyapounov-Schmidt method, Proposition 5.5 and Lemma 5.1, we have the following statement important for future use:

Corollary 5.6. *For every (σ_0, k_0), $k_0 \neq 0$, $\operatorname{Re}\sigma_0 > 0$, $\sigma_0 \in \sigma(JL(k_0))$, the set*

$$\{(\sigma, k),\, \sigma \in \sigma(J(L(k))\}$$

in a vicinity of (σ_0, k_0) is the graph of an analytic curve $k \mapsto \sigma(k)$ and $\sigma(k)$ is an eigenvalue of $JL(k)$.

6. Construction of an approximate unstable solution

Let us write the system (1.9), (1.10) under the abstract form

$$\partial_t U = \mathcal{F}(U) \tag{6.1}$$

where

$$U = \begin{pmatrix} \eta \\ \varphi \end{pmatrix}, \quad \mathcal{F}(U) = \begin{pmatrix} \eta_x + G[\eta]\varphi \\ \varphi_x - \frac{1}{2}|\nabla\varphi|^2 + \frac{1}{2}\frac{(G[\eta]\varphi + \nabla\varphi\cdot\nabla\eta)^2}{1+|\nabla\eta|^2} - \alpha\eta + \beta\nabla\cdot\big(\frac{\nabla\eta}{\sqrt{1+|\nabla\eta|^2}}\big) \end{pmatrix}.$$

We shall also use the notation $Q = (\eta_\varepsilon, \varphi_\varepsilon)$ for the solitary wave. Following the method of Grenier [16] used in our previous works [31], [32], the main ingredient in the proof of Theorem 1.4 is the construction of an approximate unstable solution of (6.1) under the form

$$U = Q + \delta U^a, \quad U^a = \sum_{j=0}^{M} \delta^j U^j. \tag{6.2}$$

To measure the regularity of the approximate solution, we introduce for $U = (\eta, \varphi)$ the "norm"

$$\|U(t)\|_{E^s}^2 = \sum_{0 \le \alpha+\beta+\gamma \le s} \|\partial_t^\alpha \partial_x^\beta \partial_y^\gamma U(t,\cdot)\|_{L^2(\mathbb{R}^2)}^2.$$

Note that since we shall work in this section with linear problems and very smooth (H^∞) solutions we do not need for the moment to emphasize some differences in the regularity of each components of U.

6.1. **Construction of U^0.** In the next proposition we first construct the leading term U^0 of the approximate solution (6.2) with a maximal growth rate.

Proposition 6.1. *There exists $U^0(t,x,y) \in \cap_{s\geq 0} E^s$ such that*

$$\partial_t U^0 = J\Lambda U^0 \tag{6.3}$$

and such that there exist an integer $m \geq 1$ and $\sigma_0 > 0$ such that for every $s \geq 0$

$$\frac{1}{c_s}\frac{e^{\sigma_0 t}}{(1+t)^{\frac{1}{2m}}} \leq \|U^0(t)\|_{E^s} \leq c_s \frac{e^{\sigma_0 t}}{(1+t)^{\frac{1}{2m}}}, \quad \forall\, t \geq 0. \tag{6.4}$$

Moreover σ_0 is such that the real part of the amplification parameter of every unstable eigenmode of (6.3) is non bigger than σ_0.

Remark 6.2. *As we shall see in the proof, we can choose U^0 under the form*

$$U^0(t,x,y) = \int_I e^{\sigma(k)t} e^{iky} U(k)(x)\, dk, \quad I = I_0 \cup -I_0\,, \tag{6.5}$$

where $I_0 \subset (0,\infty)$ is a small interval with left extremity $k_0 \neq 0$ such that $\sigma_0 = \sigma(k_0)$ and $U(k)$ is an unstable eigenmode with transverse frequency k.

Proof of Proposition 6.1. We first recall that U^0 solves (6.3) if and only if

$$V^0 = P^{-1}U^0, \quad P = \begin{pmatrix} 1 & 0 \\ -Z_\varepsilon & 1 \end{pmatrix}$$

solves

$$\partial_t V^0 = JLV^0. \tag{6.6}$$

Since the matrix P is invertible and does not depend on t, it suffices to construct a solution V^0 of (6.6) which satisfies the estimate (6.4). The first step is to find the most unstable eigenmode which solves

$$\sigma U = JL(k)U \tag{6.7}$$

i.e., we are looking for the largest σ such that σ is an eigenvalue of $JL(k)$. Thanks to Theorem 5.3, we already know that there exists $k_0 \neq 0$ such that $JL(k_0)$ has a nontrivial unstable eigenvalue. Thanks to Proposition 5.2, we also know that unstable eigenmodes must be sought only for transverse frequencies k such that $k \in [0,K]$. Moreover, for $k \in [0,K]$, we have that the amplification parameter σ of the possible unstable eigenmodes should be real and satisfy $\sigma \leq M$.

Let us assume that the unstable eigenmode given by Theorem 5.3 is such that $\sigma = \delta$. Thanks to the previous remarks, the most unstable eigenmode (i.e. with the largest σ) has to be sought in the compact set $\mathcal{R}$ of $\mathbb{R}\times\mathbb{R}$ defined by

$$\mathcal{R} \equiv \big\{(\sigma,k)\,:\,\delta/2 \leq \sigma \leq M,\ |k| \leq K\big\}.$$

Moreover, thanks to Corollary 5.6, the set $\{(\sigma,k),\, \sigma > 0,\, k \neq 0, \quad \sigma \in \sigma(JL(k)\}$ is locally the graph of an analytic curve. If we define $\Omega = \big\{k,\, \exists\,\sigma,\, \sigma > \delta/2,\, \sigma \in \sigma(JL(k))\big\}$, we thus get that Ω is a bounded (and non empty) open set of $\mathbb{R}$. One can decompose Ω as $\Omega = \cup_m I_m$ where I_m are disjoint, open and bounded intervals which are the connected components of Ω. On each I_m the above considerations prove that there exists an analytic function $k \mapsto \sigma(k)$ such that $\sigma(k)$ is the only eigenvalue of $JL(k)$ in $\sigma > 0$. We shall prove next that $k \mapsto \sigma(k)$ has a continuous extension to $\overline{I_m}$. Indeed, if k_n is a sequence converging to an extremity κ of I_m, since $\sigma(k_n)$ is bounded ($\sigma(k_n) \in \mathcal{R}$), then we can extract a subsequence not relabelled such that $\sigma(k_n)$ tends to some σ. Moreover, we also have $\sigma \geq \delta/2$, and $\sigma \in \sigma(JL(\kappa))$ since $JL(k)$ depends continuously on k. Thanks to Proposition 5.5, σ is actually an eigenvalue of $JL(\kappa)$ and hence is the only unstable eigenvalue

of $JL(\kappa)$, thanks to Lemma 5.1. By uniqueness of the limit, we get that $\lim_{k \to \kappa, k \in I_m} \sigma(k) = \sigma$ and hence, we can define a continuous function on $\overline{I_m}$. Finally, we also notice that if $\partial I_m \cap \partial I_{m'} \neq \emptyset$, then the continuations must coincide again thanks to the fact that there is at most one unstable eigenmode. Consequently, we have actually a well-defined continuous function $k \to \sigma(k)$ on $\overline{\Omega}$ which is a compact set. This allows to define k_0 and σ_0 by

$$\sigma_0 \equiv \sigma(k_0) = \sup\{\sigma(k), \quad k \in \overline{\Omega}\} > 0$$

(k_0 is not necessarily unique). Note that $k_0 \neq 0$ thanks to Lemma 5.1. Moreover, $\sigma(k)$ is an analytic function in the vicinity of k_0 and hence, there exists $m \geq 2$ so that

$$\sigma'(k_0) = \cdots = \sigma^{(m-1)}(k_0) = 0, \quad \sigma^{(m)}(k_0) \neq 0. \tag{6.8}$$

Let $I_0 \subset \Omega$ be an interval containing k_0 which does meet zero. For $k \in I_0$, let us denote by $U(k)$ the unstable mode corresponding to transverse frequency k and amplification parameter $\sigma(k)$.

Taking I_0 sufficiently small, one can take a smooth curve $k \mapsto U(k) \in H^\infty$ which is continuous from I_0 to H^s for every s. Indeed, by continuity of $k \mapsto \sigma(k)$, we can choose a disk $B(\sigma(k_0), r) \subset \{\mathrm{Re}\, \sigma > 0\}$ such that for every $k \in I_0$, $\sigma(k)$ belongs to the interior of the disk. In particular on $\partial B(\sigma(k_0), r)$, there is no eigenvalue of $JL(k)$ for $k \in I_0$ and hence thanks to Proposition 5.5, we get that the resolvent $(JL(k) - \sigma)^{-1}$ of $JL(k)$ is well defined for $(\sigma, k) \in \partial B(\sigma(k_0), r) \times \overline{I_0}$. Consequently, the eigenprojection on the only unstable eigenmode with transverse frequency k can be written under the form

$$P(k) = \frac{1}{2\pi i} \int_{\partial B(\sigma(k_0), r)} (\sigma - JL(k))^{-1} \, d\sigma.$$

This allows to choose $U(k)$ under the form

$$U(k) = \frac{1}{2\pi i} \int_{\partial B(\sigma(k_0), r)} (\sigma - JL(k))^{-1} U(k_0) \, d\sigma. \tag{6.9}$$

With this definition, $U(k)$ is non trivial for k in a vicinity of k_0 and depends smoothly on k since $JL(k)$ depends analytically on k for $k \neq 0$. We have that $\sigma(k) = \sigma(-k)$. By the definition (6.9), we also have $\overline{U(k)} = U(-k)$. Then we set $I = I_0 \cup -I_0$ and

$$V^0(t, x, y) \equiv \int_I e^{\sigma(k) t} e^{iky} \, U(k) \, dk,$$

where the dependence in x of V^0 is in $U(k)$. Note that V^0 is real-valued by the choice of I. By the Bessel-Parseval identity, we get for every s, $\alpha \in \mathbb{N}$ that

$$\|\partial_t^\alpha V^0(t, \cdot)\|_{H^s(\mathbb{R}^2)}^2 = C \int_I e^{2\sigma(k) t} \sum_{s_1 + s_2 \leq s} |\sigma(k)|^{2\alpha} k^{2 s_2} |\partial_x^{s_1} U(k)|_{L^2(\mathbb{R})}^2 \, dk,$$

where C is an harmless number. Recall that $\sigma_0 \equiv \sigma(k_0)$. Thanks to (6.8), we can apply the Laplace method (see e.g. [13, 14]) and obtain that for every s, $\alpha \geq 0$ there exists $c_{s,\alpha} \geq 1$ such that for every $t \geq 0$

$$\frac{1}{c_{s,\alpha}} \frac{1}{(1+t)^{\frac{1}{2m}}} e^{\sigma_0 t} \leq \|\partial_t^\alpha V^0(t, \cdot)\|_{H^s(\mathbb{R}^2)} \leq \frac{c_{s,\alpha}}{(1+t)^{\frac{1}{2m}}} e^{\sigma_0 t}.$$

This completes the proof of Proposition 6.1. $\square$

6.2. **Construction of U^a.** The aim of this section is to prove the following statement.

Proposition 6.3. *For every $M \geq 0$, there exists an expansion*

$$U^a = U^0 + \sum_{j=1}^{M+1} \delta^j U^j, \quad U^j \in \mathcal{C}^\infty(\mathbb{R}_+, H^\infty(\mathbb{R}^2)), \quad \delta \in \mathbb{R} \tag{6.10}$$

such that for every j, $U^j(0) = 0$ and for some $C_{s,j}$ we have the estimates

$$\|U^j(t)\|_{E^s} \leq \frac{C_{s,j}}{(1+t)^{\frac{j+1}{2m}}} e^{(j+1)\sigma_0 t}, \quad \forall\, t \geq 0. \tag{6.11}$$

Moreover, $Q + \delta U^a$ is an approximate solution of (6.1) in the sense that

$$\partial_t\big(Q + \delta U^a\big) - \mathcal{F}(Q + \delta U^a) = R^{ap} \tag{6.12}$$

and there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0]$, the estimate,

$$\|R^{ap}(t)\|_{E^s} \leq \frac{C_{M,s}\delta^{M+3}}{(1+t)^{\frac{M+3}{2m}}} e^{(M+3)\sigma_0 t}, \tag{6.13}$$

holds for $t \in [0, T^\delta]$, where T^δ is such that

$$\frac{e^{\sigma_0 T^\delta}}{(1+T^\delta)^{\frac{1}{2m}}} = \frac{1}{\delta}\,.$$

Proof of Proposition 6.3. By using the Taylor expansion of $\mathcal{F}$

$$\mathcal{F}(Q + \delta U) = \mathcal{F}(Q) + \sum_{k=1}^{M+2} \frac{\delta^k}{k!} D^k\mathcal{F}[Q]\big(U, \dots, U\big) + \delta^{M+3} R_{M,\delta}(U),$$

we can plug the expansion (6.10) into the equation (6.1) and identify the terms in front of each power of δ to get for every $j \geq 1$

$$\partial_t U^j - J\Lambda U^j = \sum_{p=2}^{j+1} \sum_{\substack{0 \leq l_1, \dots, l_p \leq M \\ l_1 + \dots + l_p = j+1-p}} \frac{1}{p!} D^p\mathcal{F}[Q]\big(U^{l_1}, \dots, U^{l_p}\big). \tag{6.14}$$

Note that the right hand-side of (6.14) involves only the U^l for $l \leq j-1$. This will allow to solve these equations by induction. Moreover, thanks to (6.5), the Fourier transform in y of U^0 is compactly supported. Consequently, it will be possible to solve the equations (6.14) with the Fourier transform of U^j compactly supported in y (in $B(0, R(|j|+1))$ for example). This remark yields the introduction of the following "norms" for functions of x only :

$$|U(t)|^2_{X^s_k} = \sum_{0 \leq \alpha + \beta \leq s} \Big(\big|\partial_t^\alpha \partial_x^\beta U_1(t, \cdot)\big|^2_{H^1(\mathbb{R})} + \big|\partial_t^\alpha \partial_x^\beta U_2\big|^2_{\dot{H}^{\frac{1}{2}}_k(\mathbb{R})} \Big),$$

with the definition

$$|\varphi|^2_{\dot{H}^{\frac{1}{2}}_k(\mathbb{R})} \equiv \Big|\frac{|D_x|}{1 + |D_x|^{\frac{1}{2}}}\varphi\Big|^2_{L^2(\mathbb{R})} + |k|^2|\varphi|^2_{L^2(\mathbb{R})}.$$

Note that the "semi-norm" $H^1 \times \dot{H}^{\frac{1}{2}}_k$ is the "energy norm" which is naturally associated to the operator $L(k)$. Note also that it does not give any control on the L^2 norm of U_2 for $k = 0$. In the sequel k will range in a compact set containing the origin. Thus we will only pay attention to the uniformness of the bounds near $k = 0$ and for that purpose $\dot{H}^{\frac{1}{2}}_k$ turns out to be quite natural. The main ingredient towards the proof of Proposition 6.3 will be the following result.

Proposition 6.4 (Semi-group bound for $J\Lambda(k)$)**.** *Let us fix $\gamma > \sigma_0$ (where σ_0 is defined in the proof of Proposition 6.1), $\rho > 0$ and $s \in \mathbb{N}$. For every every $F(t,x,k)$, $F(\cdot,\cdot,k) \in C^\infty(\mathbb{R}; H^\infty(\mathbb{R}))$ satisfying uniformly for $|k| \leq K$ the estimates*

$$\sum_{\alpha+\beta\leq s} \|\partial_t^\alpha \partial_x^\beta F(t,\cdot,k)\|_{L^2} \leq \Lambda_s \frac{e^{\gamma t}}{(1+t)^\rho}, \quad \forall\, t \geq 0, \tag{6.15}$$

if U solves

$$\partial_t U = J\Lambda(k)U + F, \quad U(0) = 0, \tag{6.16}$$

then there exists C_s depending only on Λ_{s+s_0} for some $s_0 \geq 0$ such that for every k, $|k| \leq K$,

$$|U(t,\cdot)|_{L^2} + |U(t,\cdot)|_{X_k^s} \leq C_s \frac{e^{\gamma t}}{(1+t)^\rho}, \quad \forall\, t \geq 0\,. \tag{6.17}$$

Remark 6.5. Notice that in particular (6.17) provides bounds of U and its time derivatives in usual Sobolev spaces. These bounds will be used in the application of Proposition 6.4 to the proof of Proposition 6.3.

Proof of Proposition 6.4. We shall focus on the proof of the estimate (6.17) assuming that U is a smooth solution of (6.16). We shall not detail the proof of the existence of the solution which can be obtained in a classical way (for example by using the vanishing viscosity method as in [24]) once the a priori estimates (6.17) are established. By using again the change of unknown $V = P^{-1}U$, it is equivalent to study the equation

$$\partial_t V = JL(k)V + F, \quad V(0) = 0 \tag{6.18}$$

with F satisfying the estimates (6.15) and to prove that V verifies the estimate (6.17). We shall first ignore the estimate of the L^2 norm of V_2 and prove the estimate

$$|V(t,\cdot)|_{X_k^s} \leq C_s \frac{e^{\gamma t}}{(1+t)^\rho}, \quad \forall\, t \geq 0, \quad \forall\, k,\, |k| \leq K \tag{6.19}$$

by induction on s. We start with the proof of the estimate for $s = 0$.

6.2.1. *Proof of* (6.19) *for $s = 0$.* By using the Laplace transform, we shall first reduce the proof of the estimate to a resolvent estimate. Let us fix $T > 0$ and introduce $G(t,x,k)$ such that

$$G = 0,\, t < 0, \quad G = 0,\, t > T, \quad G(t,x,k) = F(t,x,k),\, t \in [0,T].$$

We notice that the solution $\tilde{V}$ of

$$\partial_t \tilde{V} = JL(k)\tilde{V} + G, \quad \tilde{V}(0) = 0, \quad \forall\, t \geq 0 \tag{6.20}$$

verifies

$$V(\tau,x,k) = \tilde{V}(\tau,x,k), \quad \forall\, \tau \in [0,T]. \tag{6.21}$$

Indeed, $W = V - \tilde{V}$ is a solution of

$$\partial_t W = JL(k)W, \quad W(0) = 0, \quad t \in [0,T]. \tag{6.22}$$

By using again the decomposition (5.7) of $L(k)$, we get the energy estimate

$$\frac{1}{2}\frac{d}{dt}(L_0(k)W, W) = \mathrm{Re}\,\big(JL_1 W, L_0(k)W\big), \quad t \in [0,T]. \tag{6.23}$$

The right hand side was already estimated in the proof of Proposition 5.2 (see (5.11)). We have proven that

$$|\big(JL_1 W, L_0(k)W\big)| \leq C|W|_{X_k^0}^2, \tag{6.24}$$

where C is a constant independent of t, k, $|k| \leq K$ and $W \in X_k^0$. By using the estimate (3.4) we have also seen in (5.10) that for some $c > 0$

$$(L_0 W, W) \geq c|W|^2_{X_k^0}, \quad \forall\, k,\ |k| \leq K. \tag{6.25}$$

Next, we can integrate (6.23) in time and use (6.24), (6.25) to get

$$|W(t)|^2_{X_k^0} \leq C \int_0^t |W(s)|^2_{X_k^0}\, ds, \quad \forall\, t \in [0, T].$$

By the Gronwall inequality, we get that $|W(t)|^2_{X_k^0}$ vanishes on $[0, T]$. This implies that $W_1{=}0$ on $[0, T]$ and then that $W_2 = 0$ on $[0, T]$ by using the second equation of (6.22). Consequently, we shall study (6.20). For some γ_0 such that

$$\sigma_0 < \gamma_0 < \gamma, \tag{6.26}$$

let us set

$$W(\tau, x) = \mathcal{L}\tilde{V}(\gamma_0 + i\tau), \quad H(\tau, x) = \mathcal{L}G(\gamma_0 + i\tau), \quad (\tau, x) \in \mathbb{R}^2\,,$$

where $\mathcal{L}$ stands for the Laplace transform in time :

$$\mathcal{L}f(\gamma_0 + i\tau) = \int_0^\infty e^{-\gamma_0 t - i\tau\, t} f(t)\, dt.$$

Since $\tilde{V}(0) = 0$, W solves the resolvent equation

$$(\gamma_0 + i\tau)W - JL(k)W = H(\tau, \cdot). \tag{6.27}$$

By the choice of γ_0 in (6.26), $\gamma_0 + i\tau$ is not in the spectrum of $JL(k)$ for every k. Consequently, W is given by

$$W = \big((\gamma_0 + i\tau)\mathrm{Id} - JL(k)\big)^{-1} H.$$

The next step is to obtain an estimate of W uniform in τ. We first provide the bound for large values of τ (note that here we do not use that $\gamma_0 > \sigma_0$). Here is the precise statement.

Lemma 6.6. *Fix $\gamma_0 > 0$ and $K > 0$. There exist $M > 0$ and $C > 0$ such that for every $|\tau| \geq M$, every $\gamma \geq \gamma_0$ every $f = (f_1, f_2) \in H^1 \times H^{\frac{1}{2}}$, every $|k| \leq K$ if $U = (U_1, U_2)$ solves*

$$(\gamma + i\tau)U = JL(k)U + f \tag{6.28}$$

then

$$|U_1|_{H^1} + \Big|\frac{|D_x|}{1 + |D_x|^{\frac{1}{2}}} U_2\Big|_{L^2} + |k||U_2|_{L^2} \leq C|f|_{H^1 \times H^{\frac{1}{2}}}\,. \tag{6.29}$$

Proof of Lemma 6.6. We shall use Proposition 4.8. Let us set $\Phi_- = (\eta_\varepsilon^-, 0)^t$, $\Phi_0 = (\eta_\varepsilon^0, 0)^t$, we can moreover assume that Φ_- and Φ_0 are normalized in $L^2 \times L^2$. Then, every $U = (U_1, U_2)^t \in H^2 \times H^1$ can be written as

$$U = \alpha\Phi_- + \beta\Phi_0 + U^\perp, \quad (U^\perp, \Phi_-) = 0,\ (U^\perp, \Phi_0) = 0 \tag{6.30}$$

and thanks to (4.21), we have for some $c > 0$

$$(L(k)U^\perp, U^\perp) \geq c\Big(|U_1^\perp|^2_{H^1} + \Big|\frac{|D_x|}{1 + |D_x|^{\frac{1}{2}}} U_2^\perp\Big|^2_{L^2} + |k|^2 |U_2^\perp|^2_{L^2}\Big), \quad \forall\, k,\ |k| \leq K. \tag{6.31}$$

Next, if $U = (U_1, U_2)^t \in H^2 \times H^1$ is a solution of (6.28) then

$$\gamma(L(k)U, U) = \mathrm{Re}\,(f, L(k)U). \tag{6.32}$$

To estimate the right-hand side, we use that

$$|(f, L(k)U)| \leq C\Big(\big(|f_1|_{H^1} + |f_2|_{L^2}\big)\,|U_1|_{H^1} + |(f_2, G_{\varepsilon,k}U_2)| + \big|\big(f_1, (v_\varepsilon - 1)\partial_x U_2\big)\big|\Big).$$

Next, by using (3.3), and that

$$\big|(v_\varepsilon - 1)f_1, \partial_x U_2)\big| \le |(v_\varepsilon - 1)f_1|_{H^{\frac12}} |\partial_x U_2|_{H^{-\frac12}}$$

we get that

$$|(f_2, G_{\varepsilon,k}U_2)| + |\big(f_1, (v_\varepsilon - 1)\partial_x U_2)| \le C\big(|f_1|_{H^1} + |f_2|_{H^{\frac12}}\big)\Big(\Big|\frac{|D_x|}{1+|D_x|^{\frac12}}U_2\Big|_{L^2} + |k||U_2|_{L^2}\Big).$$

Consequently, we have shown that

$$|(f, L(k)U)| \le C\big(|f_1|_{H^1} + |f_2|_{H^{\frac12}}\big)\big(|U_1|_{H^1} + \Big|\frac{|D_x|}{1+|D_x|^{\frac12}}U_2\Big|_{L^2} + |k||U_2|_{L^2}\big). \tag{6.33}$$

Furthermore, using integrations by part and some crude estimates, we can estimate the left hand-side of (6.32) as follows

$$(L(k)U, U) \ge (L(k)U^\perp, U^\perp) - C\Big(|\alpha|^2 + |\beta|^2 + (|\alpha| + |\beta|)\big(|U_1^\perp|_{L^2} + \Big|\frac{|D_x|}{1+|D_x|^{\frac12}}U_2^\perp\Big|_{L^2}\big)\Big) \tag{6.34}$$

for some $C > 0$. Consequently, we can combine (6.31), (6.30) with (6.32), (6.34), (6.33) to get that

$$\begin{aligned}
&|U_1^\perp|_{H^1} + \Big|\frac{|D_x|}{1+|D_x|^{\frac12}}U_2^\perp\Big|_{L^2}^2 + k^2|U_2^\perp|_{L^2}^2 \\
&\le C\Big(|f_1|_{H^1} + |f_2|_{H^{\frac12}} + |\alpha| + |\beta|\Big)\Big(|U_1^\perp|_{H^1} + \Big|\frac{|D_x|}{1+|D_x|^{\frac12}}U_2^\perp\Big|_{L^2} + |k||U_2^\perp|_{L^2}\Big) + C(|\alpha|^2 + |\beta|^2).
\end{aligned}$$

A use of the inequality (4.26) yields

$$|U_1^\perp|_{H^1}^2 + \Big|\frac{|D_x|}{1+|D_x|^{\frac12}}U_2^\perp\Big|_{L^2}^2 + k^2|U_2^\perp|^2 \le C\big(|f|_{H^1\times H^{\frac12}}^2 + |\alpha|^2 + |\beta|^2\big). \tag{6.35}$$

We now take the $L^2 \times L^2$ scalar product of

$$(\gamma + i\tau)U = JL(k)U + f$$

with Φ_- and Φ_0 to arrive at

$$(\gamma + i\tau)\alpha = -(U, L(k)J\Phi_-) + (f, \Phi_-)$$

and

$$(\gamma + i\tau)\beta = -(U, L(k)J\Phi_0) + (f, \Phi_0)\,.$$

By using again (3.3) and the fact that Φ_0, Φ_- are smooth and fixed, we have for $i = 0, -$,

$$|(U, L(k)J\Phi_i)| \le C\Big(|U_1|_{L^2} + \Big|\frac{|D_x|}{1+|D_x|^{\frac12}}U_2\Big|_{L^2} + |k||U_2|_{L^2}\Big).$$

Therefore, we obtain that

$$(\gamma + |\tau|)|\alpha| \;\le\; C\big(|U_1|_{L^2} + \Big|\frac{|D_x|}{1+|D_x|^{\frac12}}U_2\Big|_{L^2} + |k|\,|U_2|_{L^2} + |f|_{L^2\times L^2}\big), \tag{6.36}$$

$$(\gamma + |\tau|)|\beta| \;\le\; C\big(|U_1|_{L^2} + \Big|\frac{|D_x|}{1+|D_x|^{\frac12}}U_2\Big|_{L^2} + |k|\,|U_2|_{L^2} + |f|_{L^2\times L^2}\big). \tag{6.37}$$

Combining (6.35), (6.36) and (6.37), we obtain that for $|\tau|$ sufficiently large, depending on K and γ_0, we arrive at the (6.29). This completes the proof of Lemma 6.6. $\square$

Using Lemma 6.6, we get

$$|W(\tau,\cdot)|_{X^0_k} \le C|H(\tau,\cdot)|_{H^1\times H^{\frac12}}, \quad \forall\, \tau,\, k,\, |\tau|\ge M,\, |k|\le K. \tag{6.38}$$

Next, we give the argument for $|\tau|\le M$. Since, on the compact set $\{\lambda=\gamma_0+i\tau,\, |\tau|\le M\}$, there is no spectrum of $JL(k)$ by the choice of γ_0, we get by the continuous dependence of $L(k)$ in k that the resolvent

$$\mathcal{R}(\tau,k) = \big((\gamma_0+i\tau)\mathrm{Id} - JL(k)\big)^{-1}$$

is uniformly bounded on $[-M,M]\times[-K,K]$ i.e.

$$|W(\tau)|_{H^2\times H^1} \le C|H(\tau)|_{L^2\times L^2}, \quad \forall\, \tau,\, k,\, |\tau|\le M,\, |k|\le K.$$

Consequently, we have in particular proven the uniform estimate

$$|W(\tau,\cdot)|_{X^0_k} \le C|H(\tau,\cdot)|_{H^1\times H^{\frac12}}, \quad \forall\, \tau,\, k, \quad |k|\le K. \tag{6.39}$$

By the Bessel-Parseval identity, (6.21) and (6.39), we get

$$\begin{aligned}\int_0^T e^{-2\gamma_0 t}|V(t)|^2_{X^0_k}\,dt &\le \int_0^{+\infty} e^{-2\gamma_0 t}|\tilde V(t)|^2_{X^0_k}\,dt = C\int_{\mathbb{R}} |W(\tau)|^2_{X^0_k}\,d\tau \\ &\le C\int_{\mathbb{R}} |H(\tau)|^2_{H^1\times H^{\frac12}}\,d\tau = \int_0^T e^{-2\gamma_0 t}|F(t,k)|^2_{H^1\times L^2}\,dt\end{aligned}$$

and finally thanks to (6.15), we get that there exists $C>0$ such that for every $T>0$,

$$\int_0^T e^{-2\gamma_0 t}|V(t)|^2_{X^0_k}\,dt \le C\int_0^T \frac{e^{2(\gamma-\gamma_0)t}}{(1+t)^{2\rho}}\,dt \le C\frac{e^{2(\gamma-\gamma_0)T}}{(1+T)^{2\rho}} \tag{6.40}$$

since γ_0 was fixed such that $\gamma>\gamma_0$.

To finish the proof, we can use an energy estimate for (6.18). By using again the decomposition (5.7), we get the energy estimate

$$\frac12\frac{d}{dt}e^{-2\gamma_0 t}\big(L_0(k)V,V\big) = e^{-2\gamma_0 t}\mathrm{Re}\,\big(JL_1V, L_0(k)V\big) - 2\gamma_0 e^{-2\gamma_0 t}\big(L_0(k)U,U\big) + \mathrm{Re}\,\big(F, L_0(k)V\big).$$

Since, by using an integration by parts and (3.3) we have

$$\big|(F, L_0(k)V)\big| \le C|F|_{X^0_k}|V|_{X^0_k}, \quad \big(L_0(k)U,U\big) \le C|U|^2_{X^k_0}, \tag{6.41}$$

a new use of (6.24), (6.25) and (6.15) gives

$$e^{-2\gamma_0 t}|V(t)|^2_{X^0_k} \le C\int_0^t e^{-2\gamma_0 t}|V(s)|^2_{X^k_0}\,ds + C\int_0^t \frac{e^{2(\gamma-\gamma_0)s}}{(1+s)^{2\rho}}\,ds.$$

Consequently, we can use (6.40) to get

$$e^{-2\gamma_0 t}|V(t)|^2_{X^0_k} \le C\frac{e^{2(\gamma-\gamma_0)t}}{(1+t)^{2\rho}}.$$

This ends the proof of (6.19) for $s=0$.

Remark 6.7. The argument for $|\tau|\le M$ given above is different compared to a similar analysis in our previous works [31, 32]. In [31, 32], we use an ODE argument since the linearized about a solitary wave equation may be easily reduced to an ODE. For the water waves problem such a reduction is not clear. On the other hand, it is not clear to us how to adapt the approach presented here to the case of the KP-I type equations, the problem being that the analogue of J for the KP-I type equations is ∂_x which makes the counterpart of Proposition 5.5 more difficult to establish.

6.2.2. *Proof of* (6.19) *for* $s \geq 1$. We shall use the following estimate

$$|\partial_x f|_{L^2} \leq C\Big(\Big|\frac{|D_x|}{1+|D_x|^{\frac{1}{2}}}\partial_x f\Big|_{L^2} + \Big|\frac{|D_x|}{1+|D_x|^{\frac{1}{2}}} f\Big|_{L^2}\Big). \tag{6.42}$$

One may obtain (6.42) by analysing separately the low and the high frequencies. For the proof of (6.19) for $s \geq 1$, we proceed by induction. Let us assume that (6.19) is proven for $s' \leq s-1$ i.e.

$$|V(t)|_{X_k^{s'}} \leq C\frac{e^{\gamma t}}{(1+t)^\rho}, \quad \forall\, t \geq 0,\, \forall\, k,\, |k| \leq K,\, \forall\, s',\, s' \leq s-1. \tag{6.43}$$

We have to estimate $|\partial_t^{s-i}\partial_x^i V|_{X_k^0}$ for $i \leq s$. We shall now use an induction on i. For $i = 0$, since the coefficients of $L(k)$ do not depend on time, we get that $\partial_t^s V$ solves

$$\big(\partial_t - JL(k)\big)(\partial_t^s V) = \partial_t^s F.$$

Moreover, by using the equation (6.18) and (6.15), we get that at $t = 0$

$$|\partial_t^s V(0)|_{H^l} \leq C_{s,l}, \tag{6.44}$$

where $C_{s,l}$ depends only on norms of F at $t = 0$. Thus, we get in particular that

$$W = \partial_t^s V(t) - \partial_t^s V(0)$$

solves the equation

$$\partial_t W - JL(k)W = \tilde{F}, \quad W(0) = 0$$

with a source term $\tilde{F}$ satisfying (6.15). By using the result of the previous subsection, we get

$$|W(t)|_{X_k^0} \leq C\frac{e^{\gamma t}}{(1+t)^\rho}, \quad \forall\, t \geq 0$$

and hence

$$|\partial_t^s V(t)|_{X_k^0} \leq C\frac{e^{\gamma t}}{(1+t)^\rho}, \quad \forall\, t \geq 0. \tag{6.45}$$

Now, for $j \geq 1$, let us assume that

$$|\partial_t^{s-i}\partial_x^i V|_{X_k^0} \leq C\frac{e^{\gamma t}}{(1+t)^\rho}, \quad \forall\, i \leq j-1, \quad \forall\, t \geq 0. \tag{6.46}$$

By applying $\partial_t^{s-j}\partial_x^j$ to equation (6.18), we get the equation

$$\partial_t(\partial_t^{s-j}\partial_x^j V) = J\Big(L(k)\partial_t^{s-j}\partial_x^j V + [\partial_x^j, L(k)]\partial_t^{s-j}V\Big) + \partial_t^{s-j}\partial_x^j F. \tag{6.47}$$

Thanks to Proposition 3.8 and (6.42), we easily get the estimate

$$|[\partial_x^j, L(k)]\partial_t^{s-j}V|_{L^2} \leq C\sum_{i\leq j}|\partial_t^{s-j}\partial_x^i V|_{X_k^0}.$$

Consequently, thanks to the induction assumption (6.43), we get that

$$|[\partial_x^j, L(k)]\partial_t^{s-j}V|_{L^2} \leq C\Big(|\partial_t^{s-j}\partial_x^j V|_{X_k^0} + \frac{e^{\gamma t}}{(1+t)^\rho}\Big). \tag{6.48}$$

By taking the scalar product and the real part of (6.47), against $L(k)\partial_t^{s-j}\partial_x^j V + [\partial_x^j, L(k)]\partial_t^{s-j}V$, we get thanks to (6.15) and (6.48) that

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big(\partial_t^{s-j}\partial_x^j V, L(k)\partial_t^{s-j}\partial_x^j V\big) + \mathrm{Re}\,\big(\partial_t\partial_t^{s-j}\partial_x^j V, [\partial_x^j, L(k)]\partial_t^{s-j}V\big)&\\ \leq C\Big(\frac{e^{2\gamma t}}{(1+t)^{2\rho}} + |\partial_t^{s-j}\partial_x^j F|_{L^2}|\partial_t^{s-j}\partial_x^j V|_{X_k^0} + \mathrm{Re}\,\big(\partial_t^{s-j}\partial_x^j F, L(k)\partial_t^{s-j}\partial_x^j V\big)\Big).&\end{aligned} \tag{6.49}$$

Thanks to (6.33) and (6.15), we have

$$|\big(\partial_t^{s-j}\partial_x^j F, L(k)\partial_t^{s-j}\partial_x^j V\big)| \le C\frac{e^{\gamma t}}{(1+t)^\rho}|\partial_t^{s-j}\partial_x^j V|_{X_k^0}. \tag{6.50}$$

Moreover, by using the expression of $L(k)$, we can write

$$\operatorname{Re}\big(\partial_t\partial_t^{s-j}\partial_x^j V, [\partial_x^j, L(k)]\partial_t^{s-j}V\big) = \operatorname{Re}\big(\partial_t\partial_t^{s-j}\partial_x^j V_2, [\partial_x^j, G_{\varepsilon,k}]\partial_t^{s-j}V_2\big) + R, \tag{6.51}$$

where R can be estimated by

$$\begin{aligned}|R| \le C\Big(&|\partial_t^{s-j+1}\partial_x^{j-1}V_1|_{H^1}\left(|\partial_t^{s-j}V_1|_{H^{j+1}} + |\partial_t^{s-j}\partial_x V_2|_{H^{j-1}}\right)\\ &+ \Big|\partial_t^{s-j+1}\frac{|D_x|}{1+|D_x|^{\frac12}}\partial_x^{j-1}V_2\Big|_{L^2}|V_1|_{H^{j+1}}\Big).\end{aligned} \tag{6.52}$$

Note that to get the last term above, we have used that

$$\Big|\Big(\partial_t^{s-j+1}\partial_x^j V_2, [\partial_x^j, \partial_x\big((v_\varepsilon-1)\,\cdot\,\big)]V_1\Big)\Big| \le C\Big|\frac{|D_x|}{1+|D_x|^{\frac12}}\partial_t^{s-j+1}\partial_x^{j-1}V_2\Big|_{L^2}\Big|[\partial_x^j, \partial_x\big((v_\varepsilon-1)\,\cdot\,\big)]V_1\Big|_{H^1},$$

while for the first term we used a direct commutator of differential operators estimate. By using the induction assumptions (6.43), (6.46) and the inequality (6.42), we get

$$|\partial_t^{s-j}\partial_x V_2|_{H^{j-1}} \le C\Big(\Big|\frac{|D_x|}{1+|D_x|^{\frac12}}\partial_t^{s-j}\partial_x^j V_2\Big|_{L^2} + \frac{e^{\gamma t}}{(1+t)^\rho}\Big),$$

and thus coming back to (6.52), we obtain

$$|R| \le C\Big(\frac{e^{\gamma t}}{(1+t)^\rho}|\partial_t^{s-j}\partial_x^j V|_{X_k^0} + \frac{e^{2\gamma t}}{(1+t)^{2\rho}}\Big). \tag{6.53}$$

To estimate the first term in the right-hand side of (6.51), we write

$$\big|\big(\partial_t\partial_t^{s-j}\partial_x^j V_2, [\partial_x^j, G_{\varepsilon,k}]\partial_t^{s-j}V_2\big)\big| \le C\Big|\partial_t^{s-j+1}\frac{|D_x|}{1+|D_x|^{\frac12}}\partial_x^{j-1}V_2\Big|_{L^2}\Big|[\partial_x^j, G_{\varepsilon,k}|\partial_t^{s-j}V_2\Big|_{H^{\frac12}}$$

and hence the induction assumption (6.43), (6.46) and the commutator estimate (3.30) yield

$$\big|\big(\partial_t\partial_t^{s-j}\partial_x^j V_2, [\partial_x^j, G_{\varepsilon,k}]\partial_t^{s-j}V_2\big)\big| \le C\Big(\frac{e^{\gamma t}}{(1+t)^\rho}|\partial_t^{s-j}\partial_x^j V|_{X_k^0} + \frac{e^{2\gamma t}}{(1+t)^{2\rho}}\Big). \tag{6.54}$$

Consequently, we can integrate (6.49) in time and use (6.44), (6.50), (6.51), (6.53), (6.54) to obtain

$$\big(\partial_t^{s-j}\partial_x^j V, L(k)\partial_t^{s-j}\partial_x^j V\big)(t) \le C\Big(\frac{e^{2\gamma t}}{(1+t)^{2\rho}} + \int_0^t \frac{e^{\gamma\tau}}{(1+\tau)^\rho}|\partial_t^{s-j}\partial_x^j V(\tau)|_{X_k^0}\,d\tau\Big). \tag{6.55}$$

A crude bound from below on $L(k)$ gives for some $c>0$, $C>0$,

$$\begin{aligned}&\big(\partial_t^{s-j}\partial_x^j V, L(k)\partial_t^{s-j}\partial_x^j V\big)(t)\\ &\ge c\big|\partial_t^{s-j}\partial_x^j V\big|_{X_k^0}^2 - C\Big(\big|\partial_t^{s-j}\partial_x^j V_1\big|_{L^2}^2 + \big|\partial_t^{s-j}\partial_x^j V_1\big|_{H^{\frac12}}\Big|\frac{|D_x|}{1+|D_x|^{\frac12}}\partial_t^{s-j}\partial_x^j V_2\Big|_{L^2}\Big), \quad \forall\, k,\ |k|\le K\end{aligned}$$

and hence by the interpolation inequality

$$\forall\,\delta>0,\ \exists\, C(\delta)>0:\quad \big|\partial_t^{s-j}\partial_x^j V_1\big|_{H^{\frac12}} \le \delta\big|\partial_t^{s-j}\partial_x^j V_1|_{H^1} + C(\delta)\big|\partial_t^{s-j}\partial_x^j V_1|_{L^2}$$

we get by choosing δ sufficiently small and the induction assumption (6.43) that

$$\big(\partial_t^{s-j}\partial_x^j V, L(k)\partial_t^{s-j}\partial_x^j V\big)(t) \ge \frac{c}{2}\big|\partial_t^{s-j}\partial_x^j V(t)|_{X_k^0}^2 - C\frac{e^{2\gamma t}}{(1+t)^{2\rho}}.$$

Consequently, we can plug this last estimate into (6.55) to get

$$\begin{aligned}\left|\partial_t^{s-j}\partial_x^j V(t)\right|^2_{X_k^0} &\le C\Big(\frac{e^{2\gamma t}}{(1+t)^{2\rho}}+\int_0^t \frac{e^{\gamma\tau}}{(1+\tau)^{\rho}}|\partial_t^{s-j}\partial_x^j V(\tau)|_{X_k^0}\,d\tau\Big)\\ &\le C(\delta)\frac{e^{2\gamma t}}{(1+t)^{2\rho}}+\delta\int_0^t |\partial_t^{s-j}\partial_x^j V(\tau)|^2_{X_k^0}\,d\tau\end{aligned}$$

for every $\delta > 0$. Note that we have used the inequality (4.26) to get the last estimate. By the choice $\delta < 2\gamma$, we get from the Gronwall inequality that

$$\left|\partial_t^{s-j}\partial_x^j V(t)\right|^2_{X_k^0} \le C\frac{e^{2\gamma t}}{(1+t)^{2\rho}}.$$

This ends the proof of (6.19).

6.2.3. L^2 *estimate.* To finish the proof of (6.17), it remains to estimate the L^2 norm of V_2 which is not given by the estimate (6.19) for small k. It suffices to use the equation for V_2 in (6.18) which gives that

$$|V_2(t)|_{L^2} \le C\int_0^t \big(|\partial_x V_2(\tau)|_{L^2}+|V_1(\tau)|_{H^2}+|F_2(\tau)|_{L^2}\big)d\tau$$

and then to use (6.19) (for $s = 1$) together with (6.42) and (6.15), to get

$$|V_2(t)|_{L^2} \le C\frac{e^{\gamma t}}{(1+t)^{\rho}}.$$

This ends the proof of Proposition 6.4. □

6.2.4. *End of the proof of Proposition 6.3.* We proceed by induction. We have already built U^0 in Proposition 6.1. Fix $j \ge 1$ and assume that the U^l are built for $l \le j-1$. We shall estimate the solution of (6.14) by using Proposition 6.4. Towards this, it suffices to check assumption (6.15), where the source term is defined by the right hand-side of (6.14). Let us denote by $\mathcal{S}^j(t,x,y)$ the right hand side of (6.14) and by $\hat{\mathcal{S}}^j(t,x,k)$ its Fourier transform with respect to y. From Proposition 3.9 and the standard product estimates in Sobolev spaces, we get

$$|\hat{\mathcal{S}}^j(t,\cdot,k)|_{F^s} \le C\sum_{p=2}^{j+1}\sum_{\substack{0\le l_1,\dots,l_p\le M\\ l_1+\cdots+l_p=j+1-p}}\Big(|\hat{U}^{l_1}|_{F^{s+s_0}}*\cdots*|\hat{U}^{l_p}|_{F^{s+s_0}}\Big)(t,k)\,,$$

where $*$ stands for the convolution with respect to the k variable and $|\cdot|_{F^s}$ is naturally defined as

$$|V(t,\cdot)|_{F^s} = \sum_{\alpha+\beta\le s}|\partial_t^\alpha\partial_x^\beta v(t,\cdot)|_{L^2(\mathbb{R})}.$$

Consequently, by using repeatedly the Cauchy-Schwarz inequality in the integrations defining the convolution, the fact that the $\hat{U}^i$ are compactly supported in k, and the Bessel-Plancherel identity, we get

$$|\hat{\mathcal{S}}^j(t,\cdot,k)|_{F^s} \le C(R,s,j)\sum_{p=2}^{j+1}\sum_{\substack{0\le l_1,\dots,l_p\le M\\ l_1+\cdots+l_p=j+1-p}}\|U^{l_1}\|_{E^s}\cdots\|U^{l_p}\|_{E^s}. \tag{6.56}$$

By the induction assumption, again the Bessel-Plancherel identity and the fact that $\mathcal{S}^j(t,x,k)$ is compactly supported in k, after suitable integrations in k starting from (6.56), we finally get

$$\|\mathcal{S}^j(t)\|_{E^s} \le C(R,s,j)\frac{e^{(j+1)\sigma_0 t}}{(1+t)^{\frac{j+1}{2m}}}.$$

Consequently, since $(j+1)\sigma_0 > \sigma_0$, the estimate of $\|U^j(t)\|_{E^s}$ follows thanks to Proposition 6.4. Finally, (6.13) follows from (6.11) and crude estimates in Sobolev spaces applied to $\|R_{M,\delta}(U^a)\|_{E^s}$ and the other terms involving δ^p with p at least $M+3$. This ends the proof of Proposition 6.3. $\square$

7. Proof of Theorem 1.4 (the nonlinear analysis)

Let us set $V^a = Q + \delta U^a$ where U^a is the approximate solution given by Proposition 6.3. To prove our instability result, we shall prove that we can construct a true solution U^δ of (1.9), (1.10), that we can still consider in its abstract form (6.1), up to time $T^\delta \sim \log(1/\delta)$, under the form

$$U^\delta = V^a + U, \quad U^\delta(0) = V^a(0) = Q + \delta U^0(0). \tag{7.1}$$

We therefore need to solve the equation

$$\partial_t U = \mathcal{F}(V^a + U) - \mathcal{F}(V^a) - R^{ap}, \quad t > 0, \quad U(0) = 0 \tag{7.2}$$

and obtain estimates for U. More precisely, we need to prove that the solution of (7.2) is defined on a sufficiently large interval (of size $\log(1/\delta)$) of time in order to see the linear instability and also to prove that U remains negligible in front of V^a.

The aim of the following is to prove a priori estimates for U suitable for that purpose. These estimates rely on the transformation of the system into a quasilinear form. Once these estimates are established, the result will follow by a continuation argument as in [16] provided the number M of terms in the approximate solution is chosen sufficiently large. The proof is organized as follows:

- In the next subsection, we introduce useful notations and functional spaces. Then we state the key energy estimate.
- Then in Subsection 7.3, we study the Dirichlet-Neumann operator $G[\eta^a + \eta]\varphi$. We need to track carefully the dependence of the estimates with respect to the regularity of the surface. Here we need the case that φ is in the Sobolev scale, for this part the analysis will be very close to the one of [24], but also the case that $\varphi = \varphi_\varepsilon$ is the line solitary wave and thus φ is very smooth but not in the Sobolev scale. The technically most subtle estimate is the estimate of $D_\eta G[\eta^a + \eta]\varphi_\varepsilon \cdot h$ in H^1 when h is in H^2 which is given in Proposition 7.13.
- Next, in Subsection 7.4, we derive a quasilinear form of the system by applying three space-time derivatives to the equation. We isolate a principal part of the equation and a lower order part that mostly arises from commutators and can be considered as made of semilinear terms. Subsection 7.5 is devoted to the estimates of these semi-linear terms.
- The energy estimates (which rely on the Hamiltonian structure of the linearized water-waves system) are given in Subsection 7.8.
- Subsection 7.10 is devoted to the conclusion that is the proof of the nonlinear instability.
- Finally, in Section 8, we briefly explain how we can use our a priori estimates in order to rigorously get the local existence of a smooth solution for (7.2) by using the vanishing viscosity method.

7.1. Notations and functional spaces.

Let us first introduce several notations. For $\sigma \in \mathbb{R}$, we denote by Λ^σ the Fourier multiplier on $\mathcal{S}'(\mathbb{R}^2)$ with symbol $(1+|\xi|^2)^{\sigma/2}$, $\xi \in \mathbb{R}^2$. We shall also denote by $|\nabla|$ the Fourier multiplier by $|\xi|$. For $\alpha = (\alpha_0, \alpha_1, \alpha_2) \in \mathbb{N}^3$, we shall use the notation

$$\partial^\alpha = \partial_t^{\alpha_0} \partial_x^{\alpha_1} \partial_y^{\alpha_2}.$$

Next, for $k \in \mathbb{N}$, we set

$$\langle \partial \rangle^k u = (\partial^\alpha u)_{|\alpha| \le k}, \quad \langle \nabla \rangle^k u = (\partial_x^{\alpha_1} \partial_y^{\alpha_2} u)_{\alpha_1 + \alpha_2 \le k}\,.$$

If $\|\cdot\|$ is a norm, by $\|\langle \partial \rangle^k u\|$ we denote the sum of $\|\cdot\|$ norms of all the components of $\langle \partial \rangle^k u$ (if u is a tensor an additional summation over the components of u should be added). A similar

convention shall be used for $\langle\nabla\rangle^k u$. For $k\in\mathbb{N}$ and $U(t)=(U_1(t),U_2(t))$ we define X^k by

$$\|U(t)\|_{X^k}^2=\sum_{|\alpha|\le k}\Big(\|\partial^\alpha U_1(t)\|_{H^1}^2+\|\partial^\alpha U_2(t)\|_{H^{\frac12}}^2\Big)\approx\|\langle\partial\rangle^k U_1(t)\|_{H^1}^2+\|\langle\partial\rangle^k U_2(t)\|_{H^{\frac12}}^2.$$

For $t>0$, we define the space X_t^k of functions defined on $[0,t]\times\mathbb{R}^2$ equipped with the norm

$$\|U\|_{X_t^k}=\sup_{0\le\tau\le t}\|U(\tau)\|_{X^k}\,.$$

Next, we define $\mathcal{W}^k$ by

$$\|u(t)\|_{\mathcal{W}^k}=\sum_{|\alpha|\le k}\|\partial^\alpha u(t)\|_{L^\infty(\mathbb{R}^2)}=\|\langle\partial\rangle^k u\|_{L^\infty(\mathbb{R}^2)}$$

and for $t>0$, we use the notation $\mathcal{W}_t^k$ for the space of functions defined on $[0,t]\times\mathbb{R}^2$ equipped with the norm

$$\|U\|_{\mathcal{W}_t^k}=\sup_{0\le\tau\le t}\|U(\tau)\|_{\mathcal{W}^k}\,.$$

We shall denote by $\omega(x)$ a generic continuous, positive non decreasing function on $\mathbb{R}^+$ with $\omega\ge1$. This function may change from line to line and in fact may be chosen under the form $C(1+|x|)^N$, where N may change in each appearance of ω.

Since we want to construct a solution U^δ of (6.1) under the form $U^\delta=U+V^a$ with $V^a=(\eta^a,\varphi^a)$, we shall use the following convention throughout the section: for a function or an operator $g(U)$, we set:

$$g^\delta=g(U+V^a),\quad g^a=g(V^a)\tag{7.3}$$

and thus

$$g^\delta-g^a=g(U+V^a)-g(V^a).$$

For example, we shall use the notation

$$G^\delta\varphi-G^a\varphi=G[\eta^a+\eta]\varphi-G[\eta^a]\varphi,\quad Z^\delta-Z^a=Z[V^a+U]-Z[V^a]$$

where Z is defined in Lemma 1.1 and the abstract equation (7.2) becomes

$$\partial_t U=\mathcal{F}^\delta-\mathcal{F}^a+R^{ap}.\tag{7.4}$$

7.2. **Statement of the energy estimate.** The aim of this section is to establish an a priori energy estimate for a smooth enough solution U of (7.4) defined on $[0,T]$ and satisfying the constraint

$$1-\|\eta^a(t)\|_{L^\infty}-\|\eta(t)\|_{L^\infty}>0,\quad\forall t\in[0,T]\tag{7.5}$$

where η stands for the first component of U.

Theorem 7.1. *Let $U(t)$ a smooth solution of (7.4) on $[0,T]$ satisfying (7.5). Then for $m\ge2$, $S\ge5$ and $t\in[0,T]$ we have the estimate:*

$$\|U(t)\|_{X^{m+3}}^2\le\omega\Big(\|R^{ap}\|_{X_t^{m+3}}+\|V^a\|_{\mathcal{W}_t^{m+S}}+\|U\|_{X_t^{m+3}}\Big)\times\Big(\|R^{ap}\|_{X_t^{m+3}}^2+\int_0^t\big(\|U(\tau)\|_{X^{m+3}}^2+\|R^{ap}(\tau)\|_{X^{m+3}}^2\big)d\tau\Big).$$

Of course we can replace $m+3$ by m for m larger than 5 but we decided to keep this form of the energy estimate in order to emphasize the fact that we have differentiated three times the system to quasilinearize it before performing the energy estimate.

This estimate is far from being the best one to use in terms of regularity to get well-posedness of the Cauchy problem, we are not interested here in this issue since it is not relevant for the proof of Theorem 1.4. Indeed, the smoother the involved norms are, the better the instability result is.

In the following, we shall always assume that U verifies the constraint (7.5) without making explicit reference to it. In a similar way, as soon as a Dirichlet Neumann operator $G[\zeta]$ is involved, we always assume that ζ satisfies $1 - \|\zeta\|_{L^\infty} > 0$ without recalling it.

7.3. **Preliminary estimates on the Dirichlet-Neumann operator.** In this section, we recall some useful properties of the Dirichlet-Neumann operator $G[\eta]\varphi$. The new points with respect to similar estimates in [24], [3] is the introduction of time derivatives in our estimates and the use of Schauder elliptic regularity estimates. We need to use this elliptic theory and in some cases combine it with the Sobolev regularity theory since the solitary waves do not belong to the usual Sobolev spaces on $\mathbb{R}^2$. In this section, we absolutely do not aim at giving optimal regularity estimates, we just give the one which are sufficient for the proof of Theorem 7.1.

As in Section 3, the problem will be reduced to elliptic estimates in a flat strip, consequently, for this section, it is useful to introduce the following notations.

For a function $u(t, X, z)$ defined on $[0, T] \times \mathcal{S}$ where $\mathcal{S}$ is the strip $\mathbb{R}^2 \times (0, 1)$, we set

$$D^\alpha u = \partial_t^{\alpha_0} \partial_x^{\alpha_1} \partial_y^{\alpha_2} \partial_z^{\alpha_3} u, \quad \alpha \in \mathbb{N}^4, \quad \nabla_{X,z}^\alpha u = \partial_x^{\alpha_1} \partial_y^{\alpha_2} \partial_z^{\alpha_3} u, \quad \alpha \in \mathbb{N}^3.$$

Moreover, as previously $\|\langle \nabla_{X,z} \rangle^m u\|_{L^2(\mathcal{S})}$ will stand for the sum of the $L^2(\mathcal{S})$ norm of $\nabla_{X,z}^\alpha u$ for $|\alpha| \leq m$ (and thus this norm is equivalent to the standard Sobolev norm $H^m(\mathcal{S})$ of the strip) while $\|\langle D \rangle^m u(t)\|_{L^2(\mathcal{S})}$ will stand for the sum of the $L^2(\mathcal{S})$ norms of $D^\alpha u(t)$ for $|\alpha| \leq m$. With these definitions, we have in particular that

$$\|\langle D \rangle^m u(t)\|_{L^2(\mathcal{S})} = \sum_{l=0}^{m} \|\langle \nabla_{X,z} \rangle^{m-l} \partial_t^l u(t)\|_{L^2(\mathcal{S})} \approx \sum_{l=0}^{m} \|\partial_t^l u(t)\|_{H^{m-l}(\mathcal{S})}. \tag{7.6}$$

Finally, we also set $\|u(t)\|_{\mathcal{W}^m(\mathcal{S})} = \|\langle D \rangle^m u(t)\|_{L^\infty(\mathcal{S})}$.

In this whole subsection the time variable t is only a parameter, we shall therefore omit to write down explicitly the dependence on this parameter.

Let us establish some product estimates which will be of constant use throughout this section.

Lemma 7.2. *For $m \geq 2$, $|\alpha| + |\beta| \leq m$, and $k = 0$, 1, 2, we have*

$$\begin{aligned} \|D^\alpha u \, D^\beta v\|_{H^k(\mathcal{S})} &\leq C \|\langle D \rangle^m u\|_{H^k(\mathcal{S})} \, \|\langle D \rangle^m v\|_{H^k(\mathcal{S})}, \\ \|D^\alpha u \, D^\beta v\|_{H^k(\mathcal{S})} &\leq C \|u\|_{\mathcal{W}^{m+k}(\mathcal{S})} \, \|\langle D \rangle^m v\|_{H^k(\mathcal{S})}. \end{aligned}$$

Proof of Lemma 7.2. The second estimate is obvious. Let us prove the first one. We start with the case $k = 0$. From the symmetry of the expression, it suffices to estimate $\|D^\alpha u D^\beta v\|_{L^2(\mathcal{S})}$ for $|\alpha| \leq |\beta|$. When $\alpha \neq 0$, we can use the Sobolev embedding $H^1(\mathcal{S}) \subset L^4(\mathcal{S})$ to get

$$\|D^\alpha u D^\beta v\|_{L^2(\mathcal{S})} \leq \|D^\alpha u\|_{H^1(\mathcal{S})} \|D^\beta v\|_{H^1(\mathcal{S})} \leq \|\langle D \rangle^m u\|_{L^2(\mathcal{S})} \, \|\langle D \rangle^m v\|_{L^2(\mathcal{S})},$$

since $|\alpha| \leq |\beta| \leq m - 1$. When $\alpha = 0$, we just write

$$\|u D^\beta v\|_{L^2(\mathcal{S})} \leq \|u\|_{L^\infty(\mathcal{S})} \|\langle D \rangle^m v\|_{L^2(\mathcal{S})}$$

and the result follows from the Sobolev embedding $\|u\|_{L^\infty(\mathcal{S})} \leq C \|\langle D \rangle^m u\|_{L^2(\mathcal{S})}$ when $m \geq 2$. For $k = 1$, it suffices to use the previous estimate with u and v replaced by ∇u and v or u and ∇v. The case $k = 2$ is very simple since $H^2(\mathcal{S})$ is an algebra. This ends the proof of Lemma 7.2. □

We are now able to state our first set of estimates on the Dirichlet-Neumann operator which will be intensively used in the proof of Theorem 1.4. We start with the estimates in the Sobolev framework.

Proposition 7.3. *Let us set*

$$\underline{\omega} = \omega\big(\|\langle\partial\rangle^m \eta\|_{H^{\frac{5}{2}}(\mathbb{R}^2)} + \|\eta_0\|_{\mathcal{W}^{m+3}(\mathbb{R}^2)}\big)$$

and let $m \geq 2$.

Then we have the following estimates:

- *for $\sigma = -1/2, 1/2, 1$,*

$$\|\langle\partial\rangle^m G[\eta + \eta_0]u\|_{H^\sigma} \leq \underline{\omega}\, \|\langle\partial\rangle^m u\|_{H^{\sigma+1}}. \tag{7.7}$$

- *For $n \geq l \geq 1$, $\sigma = -1/2, 1/2, 1$,*

$$\begin{aligned} &\|\langle\partial\rangle^m \big(D_\eta^n G[\eta+\eta_0]u \cdot (h_1, \cdots, h_n)\big)\|_{H^\sigma} \\ &\qquad \leq \underline{\omega} \|\langle\partial\rangle^m u\|_{H^{\sigma+1}} \big(\prod_{j=1}^{l} \|h_j\|_{\mathcal{W}^{m+3}}\big)\big(\prod_{j=l+1}^{n} \|\langle\partial\rangle^{m+1} h_j\|_{H^1}\big) \end{aligned} \tag{7.8}$$

(the first product is defined as 1 if $l = 0$ and the second product is defined as 1 if $l = n$).

- *Finally, we have the following commutator estimate*

$$\|[\partial^\alpha, G[\eta+\eta_0]](u)\|_{H^{-\frac{1}{2}}} \leq \underline{\omega}\, \|\langle\partial\rangle^{m-1} u\|_{H^{\frac{1}{2}}}, \quad \forall\, |\alpha| \leq m. \tag{7.9}$$

Remark 7.4. *The estimates that we have stated are the useful ones for the proof of Theorem 7.1. In particular, the important thing is that the dependence in η in $\underline{\omega}$ is controlled by $\|\langle\partial\rangle^{m+3}\eta\|_{H^1}$. We shall actually prove some more precise estimates. For example, we shall get that for $m \geq 2$, $|\alpha| \leq m$, we have*

$$\|\partial^\alpha \big(G[\eta+\eta_0]u\big)\|_{H^\sigma(\mathbb{R}^2)} \leq \omega\big(\|\langle\partial\rangle^m \eta\|_{H^{\sigma+1}(\mathbb{R}^2)} + \|\eta_0\|_{\mathcal{W}^{m+\sigma+\frac{3}{2}}(\mathbb{R}^2)}\big) \|\langle\partial\rangle^m u\|_{H^{\sigma+1}(\mathbb{R}^2)} \tag{7.10}$$

for $\sigma = -1/2$, $1/2$, $3/2$ and also

$$\|\partial^\alpha D_\eta G[\eta+\eta_0](u) \cdot h\|_{H^\sigma(\mathbb{R}^2)} \leq \underline{\omega} \|\langle\partial\rangle^m u\|_{H^{\sigma+1}(\mathbb{R}^2)}\, \|\langle\partial\rangle^m h\|_{H^{\sigma+1}(\mathbb{R}^2)} \tag{7.11}$$

for $\sigma = -1/2$, $1/2$.

Proof of Proposition 7.3 . We shall split the proof in various Lemmas.

As in the work by Lannes [24] an important point is to choose in the optimal way (by using a harmonic extension) with respect to the Sobolev regularity the map which flatters the domain. We shall denote by $\mathcal{S}$ the flat strip $\mathcal{S} = \mathbb{R}^2 \times (-1, 0)$.

Lemma 7.5. *Consider H which can be written as $H = \eta + \eta_0$ and satisfies $1 - \|\eta\|_{L^\infty} - \|\eta_0\|_{L^\infty} > \tilde{\kappa}$ with $\tilde{\kappa} > 0$ and $\partial^\alpha \eta \in H^s(\mathbb{R}^2)$ for $|\alpha| \leq m$, $m \geq 2$, $s \geq 1/2$, $\eta_0 \in \mathcal{W}^k(\mathbb{R}^2)$.*

Then, there exists a map $\theta : \mathcal{S} \to \mathbb{R}$ such that $\theta(X, -1) = -1$, $\theta(X, 0) = H(X)$ which can be decomposed as

$$\theta = \theta_1(\eta) + \theta_2(\eta_0) \tag{7.12}$$

with the estimates

$$\|\langle D\rangle^m \theta_1\|_{H^k(\mathcal{S})} \leq C_{m,k} \|\langle\partial\rangle^m \eta\|_{H^{k-\frac{1}{2}}(\mathbb{R}^2)}, \qquad \|\theta_2\|_{\mathcal{W}^m(\mathcal{S})} \leq C_m(1 + \|\eta_0\|_{\mathcal{W}^m(\mathbb{R}^2)}),$$

moreover there exists $\kappa > 0$ such that

$$\partial_z \theta \geq \kappa, \quad \forall\, X \in \mathbb{R}^2, \quad \forall z \in [0, 1]\,. \tag{7.13}$$

In particular the map $(X, z) \mapsto (X, \theta(X, z))$ is a diffeomorphism from the strip $\mathcal{S} = \mathbb{R}^2 \times (-1, 0)$ to $\{(X, z) \in \mathbb{R}^2 \times \mathbb{R} \,:\, -1 < z < H(X)\}$.

Remark 7.6. *As we shall see in the proof, θ_1 is linear in η and θ_2 affine in η_0, consequently, because of the decomposition* (7.12) *we also have the property that $D\theta(H)\cdot(h_1+h_2) = \theta_1(h_1) + \big(\theta_2(h_2) - z\big)$ if h_1 is in some Sobolev space and $h_2 \in \mathcal{W}^k$. Moreover, we also deduce that for $n \geq 2$, $D^n\theta = 0$.*

As we shall see below the same idea as in [24] can be used. Our situation is slightly different since the surface is made of a Sobolev part and a smooth non-decaying part while the bottom is flat. Moreover, we have taken into account the presence of time derivatives.

Proof of Lemma 7.5. Let $\tilde\theta_1$ be defined on $\mathcal{S}$ as the (well-defined) solution of the elliptic problem

$$\Delta\tilde\theta_1 = 0, \quad \tilde\theta_1(X,-1) = 0, \quad \tilde\theta_1(X,0) = \eta(X).$$

Then by standard elliptic regularity $\tilde\theta_1 \in H^{s+\frac12}(\mathbb{R}^2)$ if $\eta \in H^s(\mathbb{R}^2)$ and hence since the time is only a parameter in the problem, $\langle D\rangle^m\tilde\theta_1 \in H^k(\mathcal{S})$ if $\langle\partial\rangle^m\eta \in H^{k-\frac12}(\mathbb{R}^2)$. Observe that the dependence of $\tilde\theta_1$ with respect to η is linear. Next we consider the function θ_1 defined on $\mathcal{S}$ by $\theta_1(X,z) = (1+z)\tilde\theta_1(X,\epsilon z)$, where $\epsilon \in (0,1)$ is a small number to be fixed later. We also consider the function θ_2 defined on $\mathcal{S}$ by $\theta_2(X,z) = \eta_0(X) + (1+\eta_0(X))z$. Then the map $\theta \equiv \theta_1 + \theta_2$ satisfies the required properties, provided ϵ is small enough. Indeed

$$\partial_z\theta(X,z) = 1 + H(X) + \epsilon\, z\, \partial_z\tilde\theta_1(X,\epsilon z) + \epsilon\int_0^z \partial_z\tilde\theta_1(X,\epsilon\zeta)d\zeta.$$

Therefore for $\epsilon \ll 1$, we can achieve (7.13) since $1 - |H|_{L^\infty} \geq \tilde\kappa > 0$.

The claimed expression for the Frechet derivatives of $\theta(H)$ in Remark 7.6 follows directly from the construction. This completes the proof of Lemma 7.5. □

Remark 7.7. *Let us observe that the map θ satisfies $\partial_x\theta(X,-1) = \partial_y\theta(X,-1) = 0$, a fact which is useful in integration by parts arguments over $\mathcal{S}$.*

We next express the Dirichlet-Neumann operator in terms of a solution of a PDE defined on $\mathbb{R}^2 \times (0,1)$ with a domain flattened by the map constructed in Lemma 7.5. For $u(X)$ a given function on $\mathbb{R}^2$, if ϕ^u is defined on the domain

$$\{(X,z) \in \mathbb{R}^2 \times \mathbb{R} \,:\, -1 < z < H(X) = \eta(X) + \eta_0(X)\}$$

and is such that $\phi^u(X,\eta(X)) = u(X)$ and $\partial_z\phi^u(X,-1) = 0$ then we can define a function ψ^u on the flat domain $\mathcal{S}$ by

$$\psi^u(X,z) = \phi^u(X,\theta(X,z)), \quad (X,z) \in \mathbb{R}^2 \times [-1,0]$$

and we have that $\psi^u(X,0) = u(X)$, $\partial_z\psi^u(X,-1) = 0$. Next, if ϕ solves the problem

$$(\partial_x^2 + \partial_y^2 + \partial_z^2)\phi = F$$

on $\{-1 < z < H(X)\}$ then $\psi(X,z) = \phi(X,\theta(X,z))$ solves

$$\text{div}_{X,z}(g(X,z)\nabla_{X,z}\psi(X,z)) = \partial_z\theta(X,z)F(X,\theta(X,z)) \tag{7.14}$$

on $\mathcal{S}$, where g is defined by

$$g(X,z) \equiv \begin{pmatrix} \partial_z\theta(X,z) & 0 & -\partial_x\theta(X,z) \\ 0 & \partial_z\theta(X,z) & -\partial_y\theta(X,z) \\ -\partial_x\theta(X,z) & -\partial_y\theta(X,z) & \frac{1+(\partial_x\theta(X,z))^2+(\partial_y\theta(X,z))^2}{\partial_z\theta(X,z)} \end{pmatrix}, \quad (X,z) \in \mathcal{S}. \tag{7.15}$$

Note that our notation is slightly different from the one of Section 3.

Consequently, if ϕ^u solves

$$(\partial_x^2 + \partial_y^2 + \partial_z^2)\phi = 0, \quad X \in \mathbb{R}^2, \quad -1 < z < H(X), \quad \phi(X,H(X)) = u(X),\ \partial_z\phi(X,-1) = 0$$

then $\psi^u(X,z)=\phi^u(X,\theta(X,z))$ solves

$$\mathrm{div}_{X,z}(g(X,z)\nabla_{X,z}\psi(X,z))=0,\quad (x,z)\in\mathcal{S},\qquad \partial_z\psi(X,-1)=0,\quad \psi(X,0)=u(X)\,. \tag{7.16}$$

We observe (see Remark 7.7) that if ψ and ϕ are smooth enough, decaying at infinity in X and are such that $\phi(X,0)=0$ and $\partial_z\psi(X,-1)=0$ then

$$\int_{\mathcal{S}}\mathrm{div}_{X,z}(g(X,z)\nabla_{X,z}\psi(X,z))\phi(X,z)dXdz=-\int_{\mathcal{S}}g(X,z)\nabla_{X,z}\psi(X,z)\cdot\nabla_{X,z}\phi(X,z)dXdz\,.$$

Coming back to the definition of the Dirichlet-Neumann operator, using the Green formula and a change of variable justified by Lemma 7.5, we can infer the identity

$$(G[\eta+\eta_0](u),v)=\int_{-1}^{0}\int_{\mathbb{R}^2}g(X,z)\nabla_{X,z}\psi^u(X,z)\cdot\nabla_{X,z}\mathbf{v}(X,z)dXdz, \tag{7.17}$$

where $\mathbf{v}(X,z)$ is such that $\mathbf{v}(X,0)=v(X)$. The identity (7.17) will be used frequently in the sequel.

Let us denote by P the elliptic operator defined by

$$P\psi\equiv\mathrm{div}_{X,z}(g(X,z)\nabla_{X,z}\psi(X,z)),$$

where θ is defined by Lemma 7.5. As before, the proof of Proposition 7.3 will follow from the study of the elliptic operator P.

At first, in view of Lemma 7.5, we shall establish a useful decomposition of g with a part which has sharp Sobolev regularity and a smooth part.

Lemma 7.8. *There exists a decomposition $g=g_1+g_2$ such that we have*

$$\|\langle D\rangle^m g_1\|_{H^k(\mathcal{S})}\leq\omega\big(\|\langle\partial\rangle^m\eta\|_{H^{k+\frac12}(\mathbb{R}^2)}+\|\eta_0\|_{\mathcal{W}^{m+k+1}(\mathbb{R}^2)}\big),\quad m\geq 2,\ k=0,\,1,\,2, \tag{7.18}$$

$$\|g_2\|_{\mathcal{W}^k(\mathcal{S})}\leq\omega\big(\|\eta_0\|_{\mathcal{W}^{k+1}(\mathbb{R}^2)}\big),\quad\forall k. \tag{7.19}$$

Proof of Lemma 7.8. We set

$$g_2(X,z)\equiv\begin{pmatrix}\partial_z\theta_2(X,z) & 0 & -\partial_x\theta_2(X,z)\\ 0 & \partial_z\theta_2(X,z) & -\partial_y\theta_2(X,z)\\ -\partial_x\theta_2(X,z) & -\partial_y\theta_2(X,z) & \frac{1+(\partial_x\theta_2(X,z))^2+(\partial_y\theta_2(X,z))^2}{\partial_z\theta_2(X,z)}\end{pmatrix},\quad (X,z)\in\mathcal{S}$$

and $g_1=g-g_2$. The estimate of g_2 is an easy consequence of Lemma 7.5. Indeed, note that since $\partial_z\theta_2=1+\eta_0$, we have that

$$|\partial_z\theta_2|\geq\kappa>0. \tag{7.20}$$

Next, most of the terms arising in g_1 can also be estimated by using Lemma 7.5. The nonlinear terms can be estimated by using Lemma 7.2. For example, let us estimate

$$\frac{1}{\partial_z\theta_2}-\frac{1}{\partial_z\theta}=\frac{\partial_z\theta_1}{\partial_z\theta_2\partial_z\theta}.$$

At first, by using the second estimate of Lemma 7.2 and (7.13), (7.20), we get

$$\big\|\langle D\rangle^m\big(\frac{\partial_z\theta_1}{\partial_z\theta_2\partial_z\theta}\big)\big\|_{H^k(\mathcal{S})}\leq\omega\big(\|\theta_2\|_{\mathcal{W}^{m+k+1}(\mathcal{S})}\big)\Big(\|\langle D\rangle^m\theta_1\|_{H^{k+1}(\mathcal{S})}+\sum_{|\alpha|+|\beta|=m,\beta\neq0}\|D^\alpha\partial_z\theta_1 D^\beta\frac{1}{\partial_z\theta_1}\big\|_{H^k(\mathcal{S})}\Big).$$

Next, we can use the first estimate of Lemma 7.2 to get

$$\big\|\langle D\rangle^m\big(\frac{\partial_z\theta_1}{\partial_z\theta_2\partial_z\theta}\big)\big\|_{H^k(\mathcal{S})}\leq\omega\big(\|\theta_2\|_{\mathcal{W}^{m+k+1}(\mathcal{S})}+\|\langle D\rangle^m\theta_1\|_{H^{k+1}(\mathcal{S})}\big)\|\langle D\rangle^m\theta_1\|_{H^{k+1}(\mathcal{S})}$$

and the result follows by using Lemma 7.5. The other terms can be handled in a similar way.

This ends the proof of Lemma 7.8. □

The next step will be to study the elliptic equation $Pu = \nabla_{X,z} \cdot F$. We shall make use of the following elliptic regularity result.

Lemma 7.9. *For $m \geq 2$, and F such that $(F_3)_{/Z=-1} = 0$, then the solution of*

$$Pu = \nabla_{X,z} \cdot F, \quad (X,z) \in \mathcal{S}, \quad u(X,0) = \partial_z u(X,-1) = 0, \tag{7.21}$$

satisfies the estimate

$$\|\langle D\rangle^m u\|_{H^k(\mathcal{S})} \leq \omega\big(\|\langle D\rangle^m g_1\|_{H^{k-1}(\mathcal{S})} + \|g_2\|_{\mathcal{W}^{m+k-1}(\mathcal{S})}\big)\|\langle D\rangle^m F\|_{H^{k-1}(\mathcal{S})}, \quad k = 1,\, 2, 3. \tag{7.22}$$

Before giving the proof Lemma 7.9, we state a corollary which is our basic tool in the proof of Proposition 7.3.

Corollary 7.10. *For $m \geq 2$, and F such that $(F_3)_{/Z=-1} = 0$, then the solution of*

$$Pu = \nabla_{X,z} \cdot F, \quad (X,z) \in \mathcal{S}, \quad u(X,0) = \partial_z u(X,-1) = 0$$

satisfies the estimate

$$\|\langle D\rangle^m u\|_{H^k(\mathcal{S})} \leq \omega\big(\|\langle \partial\rangle^m \eta\|_{H^{k-\frac{1}{2}}(\mathbb{R}^2)} + \|\eta_0\|_{\mathcal{W}^{m+k}(\mathbb{R}^2)}\big)\|\langle D\rangle^m F\|_{H^{k-1}(\mathcal{S})},\ k = 1,\, 2, 3.$$

Proof of Corollary 7.10. It suffices to combine Lemma 7.9 and Lemma 7.8. $\square$

Proof of Lemma 7.9. We need to estimate $\|\partial_t^l u\|_{H^{m+k-l}}$ for $l \in [0,m]$ where H^k stands for the standard Sobolev space in the strip. We shall reason by induction on l.

When $l = 0$, the estimate of $\|u\|_{H^{m+k}}$ is the usual elliptic regularity estimate (note that thanks to Lemma 7.5, the matrix g is positive definite thus P is indeed an elliptic operator). The needed estimate was actually established in [24] Theorem 2.9. We thus already have that

$$\|u\|_{H^{m+k}} \leq \omega\big(\|g_1\|_{H^{m+k-1}} + \|g_2\|_{W^{m+k-1,\infty}}\big)\|F\|_{H^{m+k-1}}. \tag{7.23}$$

Now, let us assume that $\|\partial_t^j u\|_{H^{m+k-j}}$ is estimated for $j \leq l-1$. Then, we can apply ∂_t^l to (7.21) to get the equation

$$P\partial_t^l u = \partial_t^l \nabla_{X,z} \cdot F - \nabla_{X,z} \cdot \big([\partial_t^l, g]\nabla_{X,z} u\big), \quad (X,z) \in \mathcal{S}, \quad \partial_t^l u(X,0) = 0,\, \partial_z \partial_t^l u(X,-1) = 0.$$

Consequently, we can use again (7.23) to get that

$$\|\partial_t^l u\|_{H^{m+k-l}} \leq \omega\big(\|g_1\|_{H^{m+k-1}} + \|g_2\|_{W^{m+k-1,\infty}}\big)\big(\|\partial_t^l F\|_{H^{m-l+k-1}} + \big\|[\partial_t^l, g]\nabla_{X,z} u\big\|_{H^{m-l+k-1}}\big). \tag{7.24}$$

To estimate the last term in the right hand side, the only difficulty is to estimate the terms involving the commutator with g_1. In this case, we need to estimate $\|\nabla_{X,z}^{\gamma_1}\partial_t^{l-j} g_1\, \nabla_{X,z}^{\gamma_2}\partial_t^j \nabla_{X,z} u\|_{H^{k-1}(\mathcal{S})}$ for $j \leq l-1$ and $|\gamma_1| + |\gamma_2| \leq m-l$. As in the proof of Lemma 7.2 we find

$$\|\nabla_{X,z}^{\gamma_1}\partial_t^{l-j} g_1\, \nabla_{X,z}^{\gamma_2}\partial_t^j \nabla_{X,z} u\|_{H^{k-1}(\mathcal{S})} \leq C\|\langle D\rangle^m g_1\|_{H^{k-1}(\mathcal{S})}\, \|\partial_t^j u\|_{H^{k+m-j}(\mathcal{S})}. \tag{7.25}$$

Indeed, when $k = 3$, this estimate is straightforward since H^2 is an algebra. Let us explain the proof when $k = 1$. As in the proof of Lemma 7.2, we can write

$$\|\nabla_{X,z}^{\gamma_1}\partial_t^{l-j} g_1\, \nabla_{X,z}^{\gamma_2}\partial_t^j \nabla_{X,z} u\|_{L^2(\mathcal{S})} \leq C\|\nabla_{X,z}^{\gamma_1}\partial_t^{l-j} g_1\|_{H^1(\mathcal{S})}\, \|\nabla_{X,z}^{\gamma_2}\partial_t^j \nabla_{X,z} u\|_{H^1(\mathcal{S})}$$

and therefore (7.25) follows except when $j = 0$ and $\gamma_2 = 0$. In this case, we just write

$$\|\nabla_{X,z}^{\gamma_1}\partial_t^l g_1\, \nabla_{X,z} u\|_{L^2(\mathcal{S})} \leq C\|\langle D\rangle^m g_1\|_{L^2(\mathcal{S})}\, \|\nabla_{X,z} u\|_{L^\infty(\mathcal{S})}$$

and the result follows by Sobolev embedding since $m \geq 2$. The proof of (7.25) when $k = 2$ follows the same lines.

We can use the induction assumption and (7.24), (7.25) to conclude since $j \leq l-1$. This ends the proof of Lemma 7.9. $\square$

Let ψ^u be defined as a solution of (7.16). We have the following bounds for ψ^u.

Lemma 7.11. *For $m \geq 2$, we have the estimate,*

$$\|\langle D\rangle^m \psi^u\|_{H^k(\mathcal{S})} \leq \omega(\|\langle \partial\rangle^m \eta\|_{H^{k-\frac{1}{2}}(\mathbb{R}^2)} + \|\eta_0\|_{\mathcal{W}^{m+k}(\mathbb{R}^2)})\|\langle \partial\rangle^m u\|_{H^{k-\frac{1}{2}}(\mathbb{R}^2)}, \quad k = 1,\, 2,\, 3.$$

Note that we state here a regularity result for all the derivatives of ψ^u whereas in Lemma 3.3 we could consider only tangential derivatives and one normal derivative. This difference is important in order to get an optimal estimate in terms of the regularity of the surface.

Proof. We split ψ^u as $\psi^u = u^H + u^r$, where u^H is defined via its Fourier transform by

$$\hat{u}^H(\xi, z) = \frac{\cosh\big(|\xi|(z+1)\big)}{\cosh|\xi|}\hat{u}(\xi), \quad \xi \in \mathbb{R}^2,\, z \in (-1, 0). \tag{7.26}$$

As in the proof of Lemma 3.2, we get the bound

$$\|\langle D\rangle^m u^H\|_{H^k(\mathcal{S})} \leq C\|\langle \partial\rangle^m u\|_{H^{k-\frac{1}{2}}(\mathbb{R}^2)}. \tag{7.27}$$

Since u^r solves $P(u^r) = -P(u^H)$ with homogeneous boundary conditions, by using Corollary 7.10, we get

$$\|\langle D\rangle^m u^r\|_{H^k(\mathcal{S})} \leq \omega(\|\langle \partial\rangle^m \eta\|_{H^{k-\frac{1}{2}}(\mathbb{R}^2)} + \|\eta_0\|_{\mathcal{W}^{m+k}(\mathbb{R}^2)})\|\langle D\rangle^m \big(g\nabla_{X,z}u^H\big)\|_{H^{k-1}(\mathcal{S})}.$$

Next, by using Lemma 7.8, we can write

$$\|\langle D\rangle^m \big(g\nabla_{X,z}u^H\big)\|_{H^{k-1}(\mathcal{S})} \leq \|\langle D\rangle^m \big(g_1\nabla_{X,z}u^H\big)\|_{H^{k-1}(\mathcal{S})} + \omega(\|\eta_0\|_{\mathcal{W}^{m+k}})\|\langle D\rangle^m u^H\|_{H^k(\mathcal{S})}.$$

From Lemma 7.2 we infer

$$\|\langle D\rangle^m \big(g_1\nabla_{X,z}u^H\big)\|_{H^{k-1}(\mathcal{S})} \lesssim \|\langle D\rangle^m g_1\|_{H^{k-1}(\mathcal{S})}\|\langle D\rangle^m u^H\|_{H^k(\mathcal{S})}.$$

This yields by using (7.8)

$$\|\langle D\rangle^m u^r\|_{H^k(\mathcal{S})} \leq \omega(\|\langle \partial\rangle^m \eta\|_{H^{k-\frac{1}{2}}(\mathbb{R}^2)} + \|\eta_0\|_{\mathcal{W}^{m+k}(\mathbb{R}^2)})\|\langle D\rangle^m u^H\|_{H^k(\mathcal{S})}$$

and hence Lemma 7.11 follows by combining this estimate with (7.27). $\square$

We are now in position to give the proof of (7.7). We first prove it for $\sigma = -1/2$, $1/2$ and $3/2$. Let us write

$$\partial^\alpha = \partial_t^j \partial_X^\beta, \quad j + |\beta| = |\alpha|.$$

We need to evaluate the quantity

$$\|\partial_t^j \partial_X^\beta\big(G[\eta+\eta_0]u\big)\|_{H^\sigma} = \|\partial_t^j \partial_X^\beta \Lambda^\sigma\big(G[\eta+\eta_0]u\big)\|_{L^2}.$$

By duality, we write for $v \in \mathcal{S}(\mathbb{R}^2)$,

$$\begin{aligned}
(\partial_t^j \partial_X^\beta \Lambda^\sigma\big(G[\eta+\eta_0]u\big), v) &= (-1)^{|\beta|}\partial_t^j (G[\eta+\eta_0]u, \partial_X^\beta \Lambda^\sigma v) \\
&= (-1)^{|\beta|}\partial_t^j \int_{\mathcal{S}} g(X,z)\nabla_{X,z}(\psi^u)\cdot\nabla_{X,z}(\partial_X^\beta \Lambda^\sigma \mathbf{v})\, dXdz \\
&= \int_{\mathcal{S}} \Lambda^{\sigma+\frac{1}{2}}\partial^\alpha (g(X,z)\nabla_{X,z}\psi^u)\cdot\nabla_{X,z}\Lambda^{-\frac{1}{2}}\mathbf{v}\, dXdz,
\end{aligned}$$

where $\mathbf{v}$ is defined by

$$\mathbf{v}(x,y,z) = \frac{\cosh\Big(\sqrt{D_x^2+D_y^2}(z+1)\Big)}{\cosh\sqrt{D_x^2+D_y^2}}(v). \tag{7.28}$$

Since we have by using (7.27) that

$$\|\nabla_{X,z}\Lambda^{-\frac{1}{2}}\mathbf{v}\|_{L^2(\mathcal{S})} \leq C\|v\|_{L^2(\mathbb{R}^2)}\,,$$

we get from Cauchy-Schwarz that

$$\Big|\int_{\mathcal{S}} \Lambda^{\sigma+\frac{1}{2}}\partial^\alpha(g(X,z)\nabla_{X,z}\psi^u)\cdot\nabla_{X,z}\Lambda^{-\frac{1}{2}}\mathbf{v}\,dXdz\Big| \leq C\|\langle D\rangle^m\big(g(X,z)\nabla_{X,z}\psi^u\big)\|_{H^{\sigma+\frac{1}{2}}(\mathcal{S})}\|v\|_{L^2(\mathbb{R}^2)}$$

and thus, we can use again Lemma 7.2 and Lemma 7.8 (note that $\sigma+1/2\in\mathbb{N}$) to get that

$$\begin{aligned}&\Big|\int_{\mathcal{S}} \Lambda^{\sigma+\frac{1}{2}}\partial^\alpha(g(X,z)\nabla_{X,z}\psi^u)\cdot\nabla_{X,z}\Lambda^{-\frac{1}{2}}\mathbf{v}\,dXdz\Big|\\ &\leq C\big(\|\langle D\rangle^m g_1\|_{H^{\sigma+\frac{1}{2}}(\mathcal{S})}+\|g_2\|_{\mathcal{W}^{m+\sigma+\frac{1}{2}}(\mathcal{S})}\big)\|\langle D\rangle^m\psi^u\|_{H^{\sigma+\frac{3}{2}}(\mathcal{S})}\|v\|_{L^2(\mathbb{R}^2)}\\ &\leq \omega\big(\|\langle\partial\rangle^m\eta\|_{H^{\sigma+1}(\mathbb{R}^2)}+\|\eta_0\|_{\mathcal{W}^{m+\sigma+\frac{3}{2}}(\mathbb{R}^2)}\big)\|\langle D\rangle^m\psi^u\|_{H^{\sigma+\frac{3}{2}}(\mathcal{S})}\|v\|_{L^2(\mathbb{R}^2)}.\end{aligned}$$

Consequently, we get by using Lemma 7.11 with $k=\sigma+3/2$ to get that

$$\begin{aligned}&\Big|\int_{\mathcal{S}} \Lambda^{\sigma+\frac{1}{2}}\partial^\alpha(g(X,z)\nabla_{X,z}\psi^u)\cdot\nabla_{X,z}\Lambda^{-\frac{1}{2}}\mathbf{v}\,dXdz\Big|\\ &\qquad\qquad\leq \omega\big(\|\langle\partial\rangle^m\eta\|_{H^{\sigma+1}(\mathbb{R}^2)}+\|\eta_0\|_{\mathcal{W}^{m+\sigma+\frac{3}{2}}(\mathbb{R}^2)}\big)\|\langle\partial\rangle^m u\|_{H^{\sigma+1}(\mathbb{R}^2)}\|v\|_{L^2(\mathbb{R}^2)}\end{aligned}$$

and hence, we find that

$$\|\partial^\alpha\big(G[\eta+\eta_0]u\big)\|_{H^\sigma(\mathbb{R}^2)} \leq \omega\big(\|\langle\partial\rangle^m\eta\|_{H^{\sigma+1}(\mathbb{R}^2)}+\|\eta_0\|_{\mathcal{W}^{m+\sigma+\frac{3}{2}}(\mathbb{R}^2)}\big)\|\langle\partial\rangle^m u\|_{H^{\sigma+1}(\mathbb{R}^2)}.$$

This proves (7.7) for $\sigma=-1/2$, $1/2$ and $3/2$ and actually the refined version (7.10).

To get the H^1 estimate, it suffices to interpolate between the $H^{\frac{1}{2}}$ estimate and the $H^{\frac{3}{2}}$ estimate. More precisely, we define the linear operator A acting on the tensor $(\partial^\alpha u)_{|\alpha|\leq m}$ as $A(\partial^\alpha u)_{|\alpha|\leq m}=\big(\partial^\alpha(G[\eta+\eta_0]\cdot u)\big)_{|\alpha|\leq m}$. Since this operator maps continuously $H^{\frac{3}{2}}$ in $H^{\frac{1}{2}}$ and $H^{\frac{5}{2}}$ into $H^{\frac{3}{2}}$, it also maps continuously H^2 in H^1. This ends the proof of (7.7)

Let us now turn to the proof of the commutator estimate (7.9). For $v\in H^{\frac{1}{2}}(\mathbb{R}^2)$, we can write

$$\begin{aligned}(7.29)\quad ([\partial^\alpha, G[\eta+\eta_0]](u),v) = &\int_{\mathcal{S}} g\nabla_{X,z}(\partial^\alpha\psi^u-\psi^{\partial^\alpha u})\cdot\nabla_{X,z}\mathbf{v}\,dXdz\\ &\qquad\qquad+\int_{\mathcal{S}}[\partial^\alpha,g]\nabla_{X,z}\psi^u\cdot\nabla_{X,z}\mathbf{v}\,dXdz,\end{aligned}$$

where $\mathbf{v}$ is again defined by (7.28). We have that

$$(7.30)\qquad P(\partial^\alpha\psi^u-\psi^{\partial^\alpha u}) = [P,\partial^\alpha]\psi^u = \mathrm{div}_{X,z}([g,\partial^\alpha]\nabla_{X,z}\psi^u)$$

and moreover $\partial^\alpha\psi^u-\psi^{\partial^\alpha u}$ satisfies homogeneous boundary conditions. Multiplying (7.30) by $\partial^\alpha\psi^u-\psi^{\partial^\alpha u}$ and integrating over $\mathcal{S}$ yields

$$\|\nabla_{X,z}(\partial^\alpha\psi^u-\psi^{\partial^\alpha u})\|_{L^2(\mathcal{S})} \leq C\|[g,\partial^\alpha]\nabla_{X,z}\psi^u\|_{L^2(\mathcal{S})}\,.$$

To estimate the commutator, we need to estimate $\|\partial^\beta g_1\partial^\gamma\nabla_{X,z}\psi^u\|_{L^2(\mathcal{S})}$ for $|\beta|+|\gamma|\leq m$, $|\gamma|\neq m$. Again, when $|\gamma|\leq m-2$, we can use the Sobolev embedding $H^1(\mathcal{S})\subset L^4(\mathcal{S})$ while when $|\gamma|=m-1$, we put $\nabla^\beta g$ in L^∞. This yields (since $m\geq 2$)

$$\|\partial^\beta g\partial^\gamma\nabla_{X,z}\psi^u\|_{L^2(\mathcal{S})} \leq \omega\big(\|\langle D\rangle^m g_1\|_{H^1}+\|g_2\|_{\mathcal{W}^m}\big)\|\langle D\rangle^{m-1}\psi^u\|_{H^1(\mathcal{S})}$$

and hence, we obtain from Lemma 7.8 and Lemma 7.11 that

$$\|[g,\partial^\alpha]\nabla_{X,z}\psi^u\|_{L^2(\mathcal{S})} \leq \underline{\omega}\|\langle D\rangle^{m-1}\psi^u\|_{H^1} \leq \underline{\omega}\|\langle\partial\rangle^{m-1}u\|_{H^{\frac{1}{2}}(\mathbb{R}^2)}.$$

Consequently, we can use (7.29) and the above estimates to get from Cauchy-Schwarz that

$$|([\partial^\alpha, G[\eta+\eta_0]](u), v)| \le \underline{\omega} \|\langle\partial\rangle^{m-1} u\|_{H^{\frac12}(\mathbb{R}^2)} \|v\|_{H^{\frac12}(\mathbb{R}^2)}.$$

This proves (7.9) by duality.

Let us now turn to the proof of the bounds on the Frechet derivative of $G[\eta+\eta_0]u$. We only consider the case $n=1$, the case $n>1$ can be handled by applying a straightforward induction argument (see [3] for similar analysis). Moreover, we focus on the case $l=0$ which is the most interesting one since we need a sharp estimate with respect to the regularity of h in this case. By duality, we write for $v \in \mathcal{S}(\mathbb{R}^2)$ and $k=0,1$ (we shall take $k=\sigma+1/2$)

$$\begin{aligned} (\partial^\alpha \Lambda^k D_\eta G[\eta+\eta_0](u)\cdot h, v) &= \int_{\mathcal{S}} \Lambda^k \partial^\alpha (g\, \nabla_{X,z}(D_\eta \psi^u \cdot h)) \cdot \nabla_{X,z} \mathbf{v}\, dX dz \\ &\quad + \int_{\mathcal{S}} \Lambda^k \partial^\alpha ((D_\eta g \cdot h) \nabla_{X,z}\psi^u) \cdot \nabla_{X,z} \mathbf{v}\, dX dz \\ &\equiv J_1 + J_2, \end{aligned}$$

where $\mathbf{v}$ is defined by (7.28). Using the Cauchy-Schwarz inequality, we obtain the bound

$$J_1 \le C \|\langle D\rangle^m (g(X,z)\nabla_{X,z}(D_\eta\psi^u \cdot h))\|_{H^k(\mathcal{S})} \|v\|_{H^{\frac12}}.$$

For $k=0,1$ and $m\ge 2$, by using Lemma 7.2 and Lemma 7.8 we find

$$\begin{aligned} \|\langle D\rangle^m (g(X,z)\nabla_{X,z}(D_\eta\psi^u \cdot h))\|_{H^k(\mathcal{S})} &\le \omega\big(\|\langle D\rangle^m g_1\|_{H^k} + \|g_2\|_{\mathcal{W}^{m+k}}\big)\, \|\langle D\rangle^m (D_\eta\psi^u\cdot h)\|_{H^{k+1}(\mathcal{S})} \\ &\le \underline{\omega}\, \|\langle D\rangle^m (D_\eta\psi^u\cdot h)\|_{H^{k+1}(\mathcal{S})}. \end{aligned} \tag{7.31}$$

Next, we need to evaluate $\|\langle D\rangle^m (D_\eta\psi^u\cdot h)\|_{H^{k+1}(\mathcal{S})}$. For that purpose, we observe that $D_\eta\psi^u\cdot h$ solves the problem

$$P(D_\eta\psi^u\cdot h) = -\mathrm{div}(D_\eta g\cdot h\ \nabla_{X,z}\psi^u)$$

on $\mathcal{S}$ with homogeneous boundary conditions. Thus, using Corollary 7.10, we get in particular that

$$\|\langle D\rangle^m D_\eta\psi^u\cdot h\|_{H^{k+1}(\mathcal{S})} \le \underline{\omega}\, \|\langle D\rangle^m\, (D_\eta g\cdot h\ \nabla_{X,z}\psi^u)\|_{H^k(\mathcal{S})}. \tag{7.32}$$

To estimate the right hand side of (7.32), we use again Lemma 7.2, we find

$$\|\langle D\rangle^m\, (D_\eta g\cdot h\ \nabla_{X,z}\psi^u)\|_{H^k(\mathcal{S})} \lesssim\ \|\langle D\rangle^m \big(D_\eta g\cdot h\big)\|_{H^k(\mathcal{S})} \|\langle D\rangle^m \nabla_{X,z}\psi^u\|_{H^k(\mathcal{S})}. \tag{7.33}$$

Using Lemma 7.11, we get

$$\|\langle D\rangle^m \nabla_{X,z}\psi^u\|_{H^k(\mathcal{S})} \le \underline{\omega} \|\, \langle\partial\rangle^m u\|_{H^{k+\frac12}(\mathbb{R}^2)}. \tag{7.34}$$

In order to finish the estimate for J_1, it remains to evaluate $\|\langle D\rangle^m D_\eta g\cdot h\|_{H^k(\mathcal{S})}$ (we recall that in this situation $D_\eta g_2\cdot h = 0$). Coming back to the definition of g in terms of θ an by using Remark 7.6, we get the bound

$$\|\langle D\rangle^m \big(D_\eta g\cdot h\big)\|_{H^k(\mathcal{S})} \le \underline{\omega}\, \|\langle\partial\rangle^m h\|_{H^{k+\frac12}(\mathcal{S})}. \tag{7.35}$$

Combining the above estimates yields the bound

$$J_1 \le \underline{\omega}\, \|\langle\partial\rangle^m u\|_{H^{k+\frac12}(\mathbb{R}^2)}\, \|\langle\partial\rangle^m h\|_{H^{k+\frac12}(\mathbb{R}^2)}\, \|v\|_{H^{\frac12}(\mathbb{R}^2)}.$$

This ends the analysis of the contribution of J_1. Let us now turn to the analysis of J_2. By using again the Cauchy-Schwarz inequality, we obtain

$$J_2 \le C \|\langle D\rangle^m ((D_\eta g(X,z)\cdot h)\nabla_{X,z}\psi^u)\|_{H^k(\mathcal{S})}\ \|v\|_{H^{\frac12}(\mathbb{R}^2)}.$$

Next, by using again Lemma 7.2, we get

$$\|\langle D\rangle^m ((D_\eta g(X,z)\cdot h)\nabla_{X,z}\psi^u)\|_{H^k(\mathcal{S})} \le C \|\langle D\rangle^m D_\eta g\cdot h\|_{H^k(\mathcal{S})}\ \|\langle D\rangle^m \psi^u\|_{H^{k+1}(\mathcal{S})}$$

and then we can conclude the bound of J_2 as we did in the analysis of J_1, see (7.33), (7.34), (7.35). We have thus proven that

$$|(\partial^\alpha \Lambda^k D_\eta G[\eta+\eta_0](u)\cdot h, v)| \le \underline{\omega} \|\langle\partial\rangle^m u\|_{H^{k+\frac12}(\mathbb{R})^2} \|\langle\partial\rangle^m h\|_{H^{k+\frac12}(\mathbb{R}^2)} \|v\|_{H^{\frac12}(\mathbb{R}^2)}.$$

From this, we deduce that

$$\|\partial^\alpha D_\eta G[\eta+\eta_0](u)\cdot h\|_{H^\sigma} \le \underline{\omega} \|\langle\partial\rangle^m u\|_{H^{\sigma+1}(\mathbb{R})^2} \|\langle\partial\rangle^m h\|_{H^{\sigma+1}(\mathbb{R}^2)}.$$

for $\sigma = -1/2,\, 1/2$ with $\sigma = k - 1/2$. This yields the desired estimate for $\sigma = -1/2,\, 1/2$ and actually the refined version (7.11) stated in Remark 7.4.

It remains to study the case $\sigma = 1$. Again, we start from

$$\begin{aligned}(\partial^\alpha \Lambda\big(D_\eta G[\eta+\eta_0](u)\cdot h\big), v) &= \int_{\mathcal{S}} \Lambda \partial^\alpha (g\, \nabla_{X,z}(D_\eta \psi^u \cdot h))\cdot \nabla_{X,z}\mathbf{v}\, dX dz \\ &\quad + \int_{\mathcal{S}} \Lambda\partial^\alpha ((D_\eta g\cdot h)\nabla_{X,z}\psi^u)\cdot \nabla_{X,z}\mathbf{v}\, dX dz \\ &\equiv \tilde{J}_1 + \tilde{J}_2.\end{aligned}$$

We first obtain for $\tilde{J}_1$ that

$$\begin{aligned}\tilde{J}_1 &\le C\|\Lambda\partial^\alpha\big(g(X,z)\nabla_{X,z}(D_\eta\psi^u\cdot h)\big)\|_{H^{\frac12}(\mathcal{S})}\|\nabla_{X,z}\Lambda^{-\frac12}\mathbf{v}\|_{L^2(\mathcal{S})} \\ &\le C\|\partial^\alpha(g(X,z)\nabla_{X,z}(D_\eta\psi^u\cdot h))\|_{H^{\frac32}(\mathcal{S})}\|v\|_{L^2(\mathbb{R}^2)}\,. \end{aligned} \tag{7.36}$$

To estimate this term, we shall use the following classical lemma about products in Sobolev spaces

Lemma 7.12. *We have the estimates*

$$\|uv\|_{H^{\frac32}(\mathcal{S})} \le C_\sigma \|u\|_{\mathcal{C}^\sigma(\mathcal{S})}\|v\|_{H^{\frac32}(\mathcal{S})} \tag{7.37}$$

for every $\sigma \in (3/2, 2)$,

$$\|D^\alpha u D^\beta v\|_{H^{\frac32}(\mathcal{S})} \le C_m \|\langle D\rangle^m u\|_{H^{\frac32}(\mathcal{S})}\|\langle D\rangle^m v\|_{H^{\frac32}(\mathcal{S})} \tag{7.38}$$

for $|\alpha|+|\beta|\le m$, $m\ge 2$.

Proof. The first estimate is an easy consequence of the fact that

$$\|f\|^2_{H^{\frac12}(\mathcal{S})} = \|f\|^2_{L^2(\mathcal{S})} + \int_{\mathcal{S}}\int_{\mathcal{S}} \frac{|f(Y)-f(Y')|^2}{|Y-Y'|^4}\, dY dY'. \tag{7.39}$$

Indeed, let us first prove that

$$\|uv\|_{H^{\frac12}(\mathcal{S})} \le C_\beta \|u\|_{\mathcal{C}^\beta(\mathcal{S})}\|v\|_{H^{\frac12}(\mathcal{S})} \tag{7.40}$$

as soon as $\beta\in(1/2,1)$. From (7.39), the term involving the L^2 norm can be easily estimated, for the other term, we write

$$\begin{aligned}\|uv\|^2_{H^{\frac12}(\mathcal{S})} &\lesssim \|u\|^2_{L^\infty}\|v\|^2_{L^2} + \int_{\mathcal{S}}\int_{\mathcal{S}}\frac{|u(Y)-u(Y')|^2\,|v(Y')|^2}{|Y-Y'|^4}\, dY dY' \\ &\quad + \int_{\mathcal{S}}\int_{\mathcal{S}}\frac{|v(Y)-v(Y')|^2\,|u(Y)|^2}{|Y-Y'|^4}\, dY dY'.\end{aligned}$$

The second integral is obviously bounded by $\|u\|_{L^\infty}^2 \|v\|_{H^{\frac12}}^2$. To estimate the first one, we use that

$$\begin{aligned}\int_{\mathcal{S}} \frac{|u(Y)-u(Y')|^2}{|Y-Y'|^4}\, dY &= \int_{|Y-Y'|\geq 1} \frac{|u(Y)-u(Y')|^2}{|Y-Y'|^4}\, dY + \int_{|Y-Y'|\leq 1} \frac{|u(Y)-u(Y')|^2}{|Y-Y'|^4}\, dY \\ &\lesssim \|u\|_{L^\infty}^2 + \|u\|_{\mathcal{C}^\beta}^2\end{aligned}$$

and hence we find

$$\int_{\mathcal{S}}\int_{\mathcal{S}} \frac{|u(Y)-u(Y')|^2\, |v(Y')|^2}{|Y-Y'|^4}\, dY dY' \lesssim \|v\|_{L^2}^2\, \|u\|_{\mathcal{C}^\beta}^2.$$

This proves (7.40). To get (7.37), it suffices to use that

$$\|uv\|_{H^{\frac32}} \leq \|uv\|_{H^1} + \|u\, \nabla_{X,z} v\|_{H^{\frac12}} + \|\nabla_{X,z} u\, v\|_{H^{\frac12}}$$

and to apply (7.40) to the second and the third term.

To prove the second estimate, it suffices to consider the case $|\alpha| \leq |\beta|$. When $|\beta| \leq m-1$, we write since $H^2(\mathcal{S})$ is an algebra the crude estimate

$$\begin{aligned}\|D^\alpha u D^\beta v\|_{H^{\frac32}} &\lesssim \|D^\alpha u D^\beta v\|_{H^2} \lesssim \|D^\alpha u\|_{H^2}\, \|D^\beta v\|_{H^2} \\ &\lesssim \|\langle D\rangle^m u\|_{H^1}\, \|\langle D\rangle^m v\|_{H^1} \lesssim \|\langle D\rangle^m u\|_{H^{\frac32}}\, \|\langle D\rangle^m v\|_{H^{\frac32}}.\end{aligned}$$

When $|\beta| = m$ and thus $|\alpha| = 0$, we use (7.37) to get

$$\|u D^\beta v\|_{H^{\frac32}} \lesssim \|u\|_{\mathcal{C}^\sigma}\, \|\langle D\rangle^m v\|_{H^{\frac32}} \lesssim \|\langle D\rangle^m u\|_{H^{\frac32}} \|\langle D\rangle^m v\|_{H^{\frac32}}.$$

Note that the last estimate is a consequence of the Sobolev embedding and the fact that $m \geq 2$. This ends the proof of Lemma 7.12.

$\square$

Let us come back to the estimate (7.36) of $\tilde{J}_1$. By using (7.38), we get

$$\begin{aligned}(7.41)\quad &\|\partial^\alpha(g(X,z)\nabla_{X,z}(D_\eta \psi^u \cdot h))\|_{H^{\frac32}(\mathcal{S})} \\ &\qquad \leq C\big(\|\langle D\rangle^m g_1\|_{H^{\frac32}(\mathcal{S})} + \|g_2\|_{\mathcal{W}^{m+2}(\mathcal{S})}\big)\, \|\langle D\rangle^m \nabla_{X,z}(D_\eta \psi^u \cdot h))\|_{H^{\frac32}(\mathcal{S})}\end{aligned}$$

and thus by using Lemma 7.8, we find

$$\tilde{J}_1 \leq \underline{\omega}\, \|\langle D\rangle^m \nabla_{X,z}(D_\eta \psi^u \cdot h))\|_{H^{\frac32}(\mathcal{S})} \|v\|_{L^2(\mathbb{R}^2)}\, .$$

Next, as already observed, we have

$$P(D_\eta \psi^u \cdot h) = -\nabla_{X,z} \cdot \big(D_\eta g \cdot h\, \nabla_{X,z} \psi^u\big) := \nabla_{X,z} \cdot H.$$

From Corollary 7.10, we have in particular

$$\|\langle D\rangle^m \big(D_\eta \psi^u \cdot h\big)\|_{H^2(\mathcal{S})} \leq \omega\big(\|\langle D\rangle^m \eta\|_{H^{\frac32}(\mathbb{R}^2)} + \|\eta_0\|_{\mathcal{W}^{m+2}(\mathbb{R}^2)}\big) \|\langle D\rangle^m H\|_{H^1(\mathcal{S})}$$

and

$$\|\langle D\rangle^m \big(D_\eta \psi^u \cdot h\big)\|_{H^3(\mathcal{S})} \leq \omega\big(\|\langle D\rangle^m \eta\|_{H^{\frac52}(\mathbb{R}^2)} + \|\eta_0\|_{\mathcal{W}^{m+3}(\mathbb{R}^2)}\big) \|\langle D\rangle^m H\|_{H^2(\mathcal{S})}.$$

Consequently, we can interpolate between the two estimates to get

$$(7.42)\qquad \|\langle D\rangle^m \big(D_\eta \psi^u \cdot h\big)\|_{H^{\frac52}(\mathcal{S})} \leq \underline{\omega}\, \|\langle D\rangle^m H\|_{H^{\frac32}(\mathcal{S})}.$$

Therefore, we infer

$$\tilde{J}_1 \leq \underline{\omega}\, \|\langle D\rangle^m (D_\eta \psi^u \cdot h))\|_{H^{\frac52}(\mathcal{S})} \|v\|_{L^2(\mathbb{R}^2)} \leq \underline{\omega}\, \|\langle D\rangle^m \big(D_\eta g \cdot h\, \nabla_{X,z} \psi^u\big)\|_{H^{\frac32}(\mathcal{S})}\, \|v\|_{L^2(\mathbb{R}^2)}.$$

By using (7.38) in Lemma 7.12, we obtain

$$\|\langle D\rangle^m \big(D_\eta g \cdot h\, \nabla_{X,z} \psi^u\big)\|_{H^{\frac32}(\mathcal{S})} \leq \|\langle D\rangle^m \big(D_\eta g \cdot h\big)\|_{H^{\frac32}(\mathcal{S})}\, \|\langle D\rangle^m \nabla_{X,z} \psi^u\|_{H^{\frac32}(\mathcal{S})}.$$

Since $D_\eta g\cdot h$ has roughly the regularity of $\nabla_{X,z}\theta_1(h)$ (by using the definition of g and Remark 7.6), we find

$$\|\langle D\rangle^m\big(D_\eta g\cdot h\big)\|_{H^{\frac32}(\mathcal{S})}\le\underline{\omega}\,\|\langle D\rangle^m\theta_1(h)\|_{H^{\frac52}(\mathcal{S})}\le\underline{\omega}\,\|\langle\partial\rangle^m h\|_{H^2(\mathbb{R}^2)}\le\underline{\omega}\,\|\langle\partial\rangle^{m+1}h\|_{H^1(\mathbb{R}^2)}$$

where the intermediate estimate comes from a new application of Lemma 7.5. Finally, from Lemma 7.11, we get

$$\|\langle D\rangle^m\nabla_{X,z}\psi^u\|_{H^1(\mathcal{S})}\le\underline{\omega}\,\|\langle\partial\rangle^m u\|_{H^{\frac32}(\mathbb{R}^2)},\quad \|\langle D\rangle^m\nabla_{X,z}\psi^u\|_{H^2(\mathcal{S})}\le\underline{\omega}\,\|\langle\partial\rangle^m u\|_{H^{\frac52}(\mathbb{R}^2)}.$$

Consequently, we can interpolate between the two estimates to get

$$\|\langle D\rangle^m\nabla_{X,z}\psi^u\|_{H^{\frac32}(\mathcal{S})}\le\underline{\omega}\,\|\langle\partial\rangle^m u\|_{H^2(\mathbb{R}^2)}. \tag{7.43}$$

We thus obtain that

$$\tilde J_1\le\underline{\omega}\,\|\langle\partial\rangle^{m+1}h\|_{H^1(\mathbb{R}^2)}\|\langle\partial\rangle^m u\|_{H^2(\mathbb{R}^2)}\,\|v\|_{L^2(\mathbb{R}^2)}.$$

By the same kind of argument, we obtain a similar estimate for $\tilde J_2$ and therefore, we get

$$\|\partial^\alpha\big(G[\eta+\eta_0]u\big)\|_{H^1(\mathbb{R}^2)}\le\underline{\omega}\,\|\langle\partial\rangle^{m+1}h\|_{H^1(\mathbb{R}^2)}\|\langle\partial\rangle^m u\|_{H^2(\mathbb{R}^2)}.$$

This completes the proof of Proposition 7.3. □

We shall also need to estimate the Dirichlet-Neumann operator in the case when it acts on a smooth function that does not belong to a Sobolev space.

Proposition 7.13. *For every $m\ge0$ and $\mu\in(0,1)$, we have the estimates*

$$\|\langle\partial\rangle^m G[\eta+\eta_0]u\|_{\mathcal{C}^{1+\mu}(\mathbb{R}^2)}\le\omega\big(\|\langle\partial\rangle^{m+3}\eta\|_{H^1}+\|\langle\partial\rangle^m\eta_0\|_{\mathcal{C}^{2+\mu}}\big)\|\langle\partial\rangle^m u\|_{\mathcal{C}^{2+\mu}} \tag{7.44}$$

and

$$\begin{aligned}\|\langle\partial\rangle^m\big(D^n_\eta G[\eta+\eta_0]u\cdot\big(h_1,\cdots,h_n\big)\big)\|_{\mathcal{C}^{1+\mu}}&\le\omega\big(\|\langle\partial\rangle^{m+3}\eta\|_{H^1}+\|\langle\partial\rangle^m\eta_0\|_{\mathcal{C}^{2+\mu}}\big)\\&\quad\Big(\|\langle\partial\rangle^m h_1\|_{\mathcal{C}^{2+\mu}}\cdots\|\langle\partial\rangle^m h_n\|_{\mathcal{C}^{2+\mu}}\Big)\|\langle\partial\rangle^m u\|_{\mathcal{C}^{2+\mu}}.\end{aligned} \tag{7.45}$$

Moreover, for $n>l\ge0$,

$$\begin{aligned}\|\langle\partial\rangle^m\big(D^n_\eta G[\eta+\eta_0]u\cdot(h_1,\cdots,h_n)\big)\|_{H^1}&\le\omega\big(\|\langle\partial\rangle^{m+3}\eta\|_{H^1}+\|\eta_0\|_{\mathcal{W}^{m+3}}\big)\\&\quad\times\Big(\prod_{j=1}^{l}\|h_j\|_{\mathcal{W}^{m+3}}\Big)\Big(\prod_{j=l+1}^{n}\|\langle\partial\rangle^{m+1}h_j\|_{H^1}\Big)\|u\|_{\mathcal{W}^{m+3}}\end{aligned} \tag{7.46}$$

(the first product is considered 1 *in the case $l=0$).*

This proposition will be used when basically (η_0,u) is the solitary wave $(\eta_\varepsilon,\varphi_\varepsilon)$ and thus the way the estimates depend on the regularity of these functions is not very important for our purpose. The important fact is again that the estimates involve at most the norm $\|\langle\partial\rangle^{m+3}\eta\|_{H^1}$. The estimate (7.46) will be very useful to estimate terms like

$$\big(G[\eta+\eta_\varepsilon]-G[\eta_\varepsilon]\big)\varphi_\varepsilon=\int_0^1 D_\eta G[\eta+s\eta_\varepsilon]\varphi_\varepsilon\cdot\eta\,ds$$

since it gives an H^1 estimate of this term as soon as η is in some Sobolev space.

Proof of Proposition 7.13. For $u\in L^\infty$, we define by ψ^u the (well-defined) solution of the elliptic boundary value problem

$$\mathrm{div}_{X,z}(g(X,z)\nabla_{X,z}\psi(X,z))=0,\quad (X,z)\in\mathcal{S},\qquad \partial_z\psi(X,-1)=0,\quad \psi(X,0)=u(X). \tag{7.47}$$

The existence of (weak) solutions of (7.47) can be obtained by using the L^∞ a-priori bound coming from the maximum principle (see [15], Chapters 2-6). Observe that thanks to the homogeneous boundary condition on $z=-1$ the maximum of ψ is necessarily reached on the boundary $z=0$. One may also obtain the well-posedness of (7.47), by Sobolev type arguments. Namely, one may approach the problem on $\mathcal{S}$ by problems on compact domains where the maximum principle holds, get solutions on these domains by the Sobolev theory and then pass to the limit by using the uniform L^∞ estimate.

The next step is to obtain regularity estimates for ψ^u. This will be a consequence of the following elliptic regularity result:

Lemma 7.14. *For $m \geq 0$ and $\mu \in (0,1)$, the L^∞ solution of*

$$Pu = F, \quad (X,z) \in \mathcal{S}, \quad u(X,0) = 0,\, \partial_z u(X,-1) = 0$$

satisfies the estimate

$$\|\langle D\rangle^m u\|_{\mathcal{C}^{2+\mu}(\mathcal{S})} \leq \omega\big(\|\langle\partial\rangle^{m+3}\eta\|_{H^1(\mathbb{R}^2)} + \|\langle\partial\rangle^m \eta_0\|_{\mathcal{C}^{2+\mu}(\mathbb{R}^2)}\big)\|\langle D\rangle^m F\|_{\mathcal{C}^\mu(\mathcal{S})}\,.$$

Proof. When no spatial derivatives are involved, we have the following classical Schauder elliptic regularity result for u (we refer to [15] for example):

$$\|u\|_{\mathcal{C}^{2+k+\mu}(\mathcal{S})} \leq \omega\big(\|g\|_{\mathcal{C}^{1+k+\mu}(\mathcal{S}))}\big)\|F\|_{\mathcal{C}^{k+\mu}(\mathcal{S})}$$

for every integer k. By an induction on the number of time derivatives involved, we easily deduce from this estimate that

$$\|\langle D\rangle^m u\|_{\mathcal{C}^{2+\mu}(\mathcal{S})} \leq \omega\big(\|\langle D\rangle^m g\|_{\mathcal{C}^{1+\mu}(\mathcal{S}))}\big)\|\langle D\rangle^m F\|_{\mathcal{C}^\mu(\mathcal{S})}. \tag{7.48}$$

To conclude, we first notice from the definition of g that

$$\|\langle D\rangle^m g\|_{\mathcal{C}^{1+\mu}(\mathcal{S}))} \leq \omega\big(\|\langle D\rangle^m \theta_1\|_{\mathcal{C}^{2+\mu}(\mathcal{S}))} + \|\langle D\rangle^m \theta_2\|_{\mathcal{C}^{2+\mu}(\mathcal{S}))}\big). \tag{7.49}$$

From the explicit expression of θ_2, we obviously have

$$\|\langle D\rangle^m \theta_2\|_{\mathcal{C}^{2+\mu}(\mathcal{S})} \leq C\|\langle\partial\rangle^m \eta_0\|_{\mathcal{C}^{2+\mu}(\mathbb{R}^2)} \tag{7.50}$$

and moreover, by Sobolev embedding and Lemma 7.5, we have for every $s > 3/2$

$$\|\langle D\rangle^m \theta_1\|_{\mathcal{C}^{2+\mu}(\mathcal{S})} \leq C\|\langle D\rangle^m \theta_1\|_{H^{2+s+\mu}(\mathcal{S})} \leq C\|\langle\partial\rangle^m \eta\|_{H^{2+s+\mu-\frac{1}{2}}(\mathbb{R}^2)} \leq C\|\langle\partial\rangle^m \eta\|_{H^4(\mathbb{R}^2)} \tag{7.51}$$

since one can always choose s sufficiently close to $3/2$ to have $2+\mu+s-1/2 < 4$. This ends the proof of Lemma 7.14.

□

We can now estimate ψ^u.

Lemma 7.15. *For every $m \geq 0$ and $\mu \in (0,1)$, we have the estimate*

$$\|\langle D\rangle^m \psi^u\|_{\mathcal{C}^{2+\mu}(\mathcal{S})} \leq \omega\big(\|\langle\partial\rangle^{m+3}\eta\|_{H^1(\mathbb{R}^2)} + \|\langle\partial\rangle^m \eta_0\|_{\mathcal{C}^{2+\mu}(\mathbb{R}^2)}\big)\|\langle\partial\rangle^m u\|_{\mathcal{C}^{2+\mu}(\mathbb{R}^2)}\,.$$

Proof. Again we consider the splitting $\psi^u = u^H + u^r$, where u^H is defined by (7.26). By standard properties of Fourier multipliers in Hölder spaces, we get

$$\|\langle D\rangle^m u^H\|_{\mathcal{C}^s(\mathcal{S})} \leq \|\langle\partial\rangle^m u\|_{\mathcal{C}^s(\mathbb{R}^2)} \tag{7.52}$$

for every $s \geq 0$ which is not an integer. Next, since u^r solves the elliptic equation $Pu^r = -Pu^H$ with homogeneous boundary conditions, we get by using Lemma 7.14 that

$$\|\langle D\rangle^m u^r\|_{\mathcal{C}^{2+\mu}(\mathcal{S})} \leq \omega\big(\|\langle\partial\rangle^{m+3}\eta\|_{H^1(\mathbb{R}^2)} + \|\langle\partial\rangle^m \eta_0\|_{\mathcal{C}^{2+\mu}(\mathbb{R}^2)}\big)\|\langle D\rangle^m Pu^H\|_{\mathcal{C}^\mu(\mathcal{S})}.$$

Furthermore, since we have

$$\|\langle D\rangle^m Pu^H\|_{\mathcal{C}^\mu(\mathcal{S})} \leq \omega\big(\|\langle D\rangle^m g\|_{\mathcal{C}^{1+\mu}(\mathcal{S})}\big)\|\langle D\rangle^m u^H\|_{\mathcal{C}^{2+\mu}(\mathcal{S})},$$

we get the claimed estimate by using (7.49), (7.50), (7.51) and (7.52). $\square$

After these preliminaries, we can get (7.44). Indeed, observe that in terms of ψ^u, the Dirichlet-Neumann operator reads

$$(G[\eta+\eta_0]u)(X) = \frac{1+(\partial_x\theta(X,0))^2+(\partial_y\theta(X,0))^2}{\partial_z\theta(X,0)}\partial_z\psi^u(X,0) - \nabla_X\theta(X,0)\cdot\nabla_X\psi^u(X,0)$$

with $\theta(X,0)=\eta+\eta_0$. Consequently, we get

$$\|\langle\partial\rangle^m G[\eta+\eta_0]u\|_{\mathcal{C}^{1+\mu}(\mathbb{R}^2)} \leq \omega\big(\|\langle\partial\rangle^m\theta\|_{\mathcal{C}^{2+\mu}(\mathcal{S})}\big)\|\langle\partial\rangle^m\psi^u\|_{\mathcal{C}^{2+\mu}(\mathcal{S})}$$

and hence (7.44) is a consequence of Lemma 7.15 and (7.50), (7.51).

The proof of (7.45) can be obtained in the same way. This is left to the reader.

Let us finally give the proof of (7.46). We shall only give the proof for $n=1$ since the argument for $n>1$ follows by a direct induction argument and we focus on the case that h is in a Sobolev space since it is the one for which we really need optimal regularity. The case that h is in an Hölder space is covered by (7.45).

Coming back to (7.17), we obtain

$$\begin{aligned}(\partial^\alpha\Lambda^{\frac{3}{2}}D_\eta G[\eta+\eta_0](u)\cdot h, v) &= \int_{\mathcal{S}}\partial^\alpha\Lambda^{\frac{3}{2}}(g\,\nabla_{X,z}(D_\eta\psi^u\cdot h))\cdot\nabla_{X,z}\mathbf{v}dXdz \\ &\quad+\int_{\mathcal{S}}\partial^\alpha\Lambda^{\frac{3}{2}}((D_\eta g\cdot h)\nabla_{X,z}\psi^u)\cdot\nabla_{X,z}\mathbf{v}\,dXdz \\ &\equiv J_1+J_2,\end{aligned}$$

where $\mathbf{v}$ is defined by (7.28) with $v\in H^{1/2}$. Using the Cauchy-Schwarz inequality, we get

$$J_1\leq C\|\langle\partial\rangle^m\Lambda^{\frac{3}{2}}(g\,\nabla_{X,z}(D_\eta\psi^u\cdot h))\|_{L^2}\|v\|_{H^{\frac{1}{2}}}\,.$$

Next (see (7.41)), we can write

$$\|\langle\partial\rangle^m\Lambda^{\frac{3}{2}}(g\,\nabla_{X,z}(D_\eta\psi^u\cdot h))\|_{L^2}\leq\omega\big(\|\langle\partial\rangle^m\eta\|_{H^2}+\|\eta_0\|_{\mathcal{W}^{m+3}}\big)\|\langle\partial\rangle^m\Lambda^{\frac{3}{2}}(\nabla_{X,z}(D_\eta\psi^u\cdot h))\|_{L^2}\,.$$

Recall that $D_\eta\psi^u\cdot h$ solves the problem

$$P(D_\eta\psi^u\cdot h) = -\mathrm{div}(D_\eta g\cdot h\;\nabla_{X,z}\psi^u)$$

on $\mathcal{S}$ with homogeneous boundary conditions. By using (7.42), we infer

$$\|\langle\partial\rangle^m\Lambda^{\frac{3}{2}}\nabla_{X,z}(D_\eta\psi^u\cdot h)\|_{L^2(\mathcal{S})}\leq\omega\big(\big(\|\langle\partial\rangle^m\eta\|_{H^{\frac{5}{2}}}+\|\eta_0\|_{\mathcal{W}^{m+3}}\big)\|\langle D\rangle^m\Lambda^{\frac{3}{2}}\big(D_\eta g\cdot h\,\nabla_{X,z}\psi^u\big)\|_{L^2(\mathcal{S})}\,.$$

Using Lemma 7.12, we obtain

$$\|\langle D\rangle^m\Lambda^{\frac{3}{2}}\big((D_\eta g\cdot h\,\nabla_{X,z}\psi^u\big)\|_{L^2(\mathcal{S})}\leq C\|\langle D\rangle^m\nabla_{X,z}\psi^u\|_{C^\sigma(\mathcal{S})}\|\langle D\rangle^m\Lambda^{\frac{3}{2}}(D_\eta g\cdot h)\|_{L^2(\mathcal{S})},$$

provided $\sigma>3/2$. Coming back to the definition of g, thanks to Lemma 7.5 and Remark 7.6, we get

$$\||\langle D\rangle^m\Lambda^{\frac{3}{2}}(D_\eta g\cdot h)\|_{L^2(\mathcal{S})}\leq\omega\big(\big(\|\langle\partial\rangle^m\eta\|_{H^{\frac{5}{2}}}+\|\eta_0\|_{\mathcal{W}^{m+3}}\big)\,\|\langle\partial\rangle^{m+1}h\|_{H^1(\mathbb{R}^2)}.$$

Next, using Lemma 7.15 with $\mu=\sigma-1$, we get

$$\|\langle D\rangle^m\nabla_{X,z}\psi^u\|_{C^\sigma(\mathcal{S})}\leq\omega\big(\|\langle\partial\rangle^{m+3}\eta\|_{H^1}+\|\langle\partial\rangle^m\eta_0\|_{\mathcal{W}^{m+3}}\big)\|\langle\partial\rangle^m u\|_{\mathcal{C}^{2+\mu}}\,.$$

Collecting the above bounds, we arrive at

$$J_1\leq\omega\big(\|\langle\partial\rangle^{m+3}\eta\|_{H^1}+\|\langle\partial\rangle^m\eta_0\|_{\mathcal{W}^{m+3}}\big)\|\langle\partial\rangle^{m+1}h\|_{H^1}\|\langle\partial\rangle^m u\|_{\mathcal{C}^{2+\mu}}\|v\|_{H^{\frac{1}{2}}}\,.$$

The estimate for J_2 is very similar and thus will be omitted. This completes the proof of Proposition 7.13. Note that we get a slightly better result than stated. $\square$

In our energy estimates, we shall also use the following lemma.

Lemma 7.16 (see Proposition 3.4 of [3])**.** *There exists $c>0$ such that for every $\eta \in W^{1,\infty}(\mathbb{R}^2)$ with $1-\|\eta\|_{L^\infty} \geq \delta$ for some $\delta > 0$ we have*

$$(G[\eta]v, v) \geq c(1+\|\eta\|_{W^{1,\infty}(\mathbb{R}^2)})^{-2}\Big\|\frac{|\nabla|}{(1+|\nabla|)^{\frac{1}{2}}}v\Big\|^2_{L^2(\mathbb{R}^2)} \quad \forall v \in H^{\frac{1}{2}}(\mathbb{R}^2)$$

and

$$\big(G[\eta]v, w\big) \leq \omega(\|\eta\|_{W^{1,\infty}(\mathbb{R}^2)})\|v\|_{H^{\frac{1}{2}}(\mathbb{R}^2)}\|w\|_{H^{\frac{1}{2}}(\mathbb{R}^2)}, \quad \forall v,\, w \in H^{\frac{1}{2}}(\mathbb{R}^2).$$

Note that we have given previously the proof of (3.4) (3.3) which are very close estimates.

7.4. **Derivation of the quasilinear form.** The aim of this subsection is to isolate a principal part which behaves as a quasilinear symmetrizable hyperbolic like system and a remainder which behaves as a semi-linear term after applying a sufficient amount of derivatives to the equation. Let us explain more precisely the strategy. We can consider the water waves system under an abstract form

$$\partial_t U = \mathcal{F}(U).$$

When applying the operator ∂^α, for $|\alpha|$ to be chosen, to the system, we find

$$\partial_t \partial^\alpha U = J\Lambda[U]\cdot \partial^\alpha U + \mathcal{R}(U) \tag{7.53}$$

where $\partial_t - J\Lambda[U]$ is the linearized equation about U and $\mathcal{R}(U)$ involves some lower order commutators. Let us set $U_\alpha = \partial^\alpha U$. If we consider only the principal part in the equation for U_α, in view of the skew symmetry of J and the symmetry of $\Lambda[U]$, one expects to get energy estimates by taking the scalar product of the equation with $\Lambda[U]U_\alpha$ and then by reiterating the same process for higher order derivatives of U_α. The energy norm associated to $\Lambda[U]$ will be the X^0 norm and thus, we expect to control the norm $\|U_\alpha(t)\|_{X^k}$.

A good "quasilinear structure" for $\partial^\alpha U$ which easily yields an energy estimate arises for (7.53) if :

i) the norm of the remainder $\|\langle\partial\rangle^k\mathcal{R}(U)\|_{X^0}$ can be estimated in terms of $\|U\|_{X^{k+|\alpha|}}$ when k is sufficiently large. In this case, we shall say that this term behaves as a semilinear term;
ii) the estimate of the commutator $[\langle\partial\rangle^k, J\Lambda[U]]V$ in the energy space X^0 involves at most the norm $\|U\|_{X^{k+|\alpha|}}$ when k is sufficiently large.

If Λ were a first order operator (this arises classically for the usual quasilinear wave equation rewritten as a first order system), by using the above second property, we expect the commutator estimate

$$\|[\langle\partial\rangle^k, J\Lambda[U]]V\|_{X^0} \lesssim \|U\|_{X^{k+|\alpha|}}\|V\|_{X^k}$$

which allows to get an a priori estimate for U under the form

$$\|U(t)\|_{X^{k+|\alpha|}} \lesssim \|U(0)\|_{X^{k+|\alpha|}} + \int_0^t \|U(\tau)\|^2_{X^{k+|\alpha|}}\, d\tau.$$

This is a good without loss estimate which can be easily combined with an approximation argument (for example the vanishing viscosity method) in order to get a local existence result and to prove Theorem 1.4 when considering a perturbation of V^a by using the Gronwall lemma. Note that in this situation, there is no need to use time and space derivatives simultaneously.

As already pointed out in the introduction, in our situation (which is formally close to the one of higher order wave equations), in order to close the energy estimate, the commutator $[\langle\partial\rangle^k, J\Lambda[U]]V$ cannot be considered as harmless (or semilinear) since its X^0 norm involves an X^m norm of V with $m>k$. Moreover, for the same reason, the term $\mathcal{R}$ in (7.53) cannot contain only semi-linear terms.

Nevertheless, the above considerations can be generalized to our framework. The equation for $\partial^\alpha U$ can be written under the following more precise form:

$$\partial_t \partial^\alpha U = J\Big(\Lambda[U]\cdot\partial^\alpha U + \mathcal{Q}[U]\cdot(\partial^\beta U)_{|\beta|\le|\alpha|}\Big) + \mathcal{R}(U) \tag{7.54}$$

where $\mathcal{Q}(U)\cdot$ is a linear operator acting on the tensor $(\partial^\beta U)_{|\beta|\le|\alpha|}$ which is of lower order than Λ but of too high order to be incorporated in the semilinear terms. We shall prove that Λ and $\mathcal{R}$ match the above properties i) and ii) for $|\alpha| = 3$. The energy estimate for (7.54) will then be obtained by proving that the subprincipal term $\mathcal{Q}$, and the higher order part of the commutators $\big[\partial^\beta, \Lambda[U]\big]$ can be incorporated as harmless lower order terms in the energy. For this argument, it is important to use space and time derivatives simultaneously.

7.4.1. *Analysis of* (6.1). Let us first denote by $U = (\eta, \varphi)$ a solution of (6.1). We first focus on the first equation of (6.1) which reads

$$\partial_t \eta = \partial_x \eta + G[\eta]\varphi. \tag{7.55}$$

Let $\mathcal{I} = \{t, x, y\}$. We first notice that for $k \in \mathcal{I}$, $\partial_k \eta$ solves the equation

$$\partial_t \partial_k \eta = \partial_x \partial_k \eta + G[\eta]\partial_k \varphi + DG[\eta]\varphi\cdot\partial_k\eta \tag{7.56}$$

and thus by using Lemma 1.1, we find

$$\partial_t \partial_k \eta = \partial_x \partial_k \eta + G(\partial_k\varphi - Z\partial_k\eta) - \nabla\cdot(\partial_k\eta\, v), \tag{7.57}$$

where we shall use for short hands throughout this section the notation

$$G = G[\eta], \quad v = v[\eta,\varphi] = \nabla\varphi - Z\nabla\eta, \quad Z = Z[\eta,\varphi] = \frac{G[\eta]\varphi + \nabla\eta\cdot\nabla\varphi}{1+|\nabla\eta|^2}$$

(the notation for Z was already introduced in Lemma 1.1). Next, as in [21], we can derive an equation for $\partial_{ijk}\eta$ for i, j, $k \in \mathcal{I}$ by applying two more derivatives to (7.56). We find

$$\partial_t\partial_{ijk}\eta = \partial_x\partial_{ijk}\eta + G\partial_{ijk}\varphi - G(Z\partial_{ijk}\eta) - \nabla\cdot(v\partial_{ijk}\eta) + \mathcal{Q}_1^{ijk}[\eta,\varphi] + \mathcal{R}_1^{ijk}[\eta,\varphi], \tag{7.58}$$

where

$$\mathcal{Q}_1^{ijk}[\eta,\varphi] = \sum_\sigma D_\eta G[\eta]\partial_{\sigma(i)\sigma(j)}\varphi\cdot\partial_{\sigma(k)}\eta \tag{7.59}$$

the sum being taken on the circular permutations σ of the set $\{i, j, k\}$ is the subprincipal part of the equation which must be handled with some care and $\mathcal{R}_1^{ijk}$ is under the form

$$\mathcal{R}_1^{ijk}[\eta,\varphi] = \sum D^n_\eta G[\eta]\partial^\gamma\varphi\cdot\big(\partial^{\beta_1}\eta, \cdots, \partial^{\beta_n}\eta\big) \tag{7.60}$$

where the sum is taken on indices $n \in \mathbb{N}^*$, $\beta_i \in \mathbb{N}^3$, $\gamma \in \mathbb{N}^3$ which verify

$$1 \le n \le 3, \quad |\beta_1| + \cdots + |\beta_n| + |\gamma| = 3, \quad |\gamma| \le 1, \quad 1 \le |\beta_i| < 3, \forall i. \tag{7.61}$$

We shall prove below that this term behaves as a semi-linear term and thus it is not necessary to write down a more precise formula for it.

Let us now study the second equation of (6.1). If we apply the operator ∂_k to the second equation of (6.1), we get by using the previous notation

$$\partial_t\partial_k\varphi = \partial_x\partial_k\varphi - v\cdot\nabla\partial_k\varphi + ZG\partial_k\varphi + ZD_\eta G\varphi\cdot\partial_k\eta + Zv\cdot\nabla\partial_k\eta + \beta\nabla\cdot\big(A(\nabla\eta)\nabla\partial_k\eta\big) - \alpha\partial_k\eta, \tag{7.62}$$

where the matrix $A(V)$ is given by

$$A(V) = \frac{\mathrm{Id}}{(1+|V|^2)^{\frac12}} - \frac{V\otimes V}{(1+|V|^2)^{\frac32}}.$$

We now find that $\partial_{ijk}\varphi$ solves

$$
(7.63)\quad \begin{aligned} \partial_t\partial_{ijk}\varphi = \partial_x\partial_{ijk}\varphi - v\cdot\nabla\partial_{ijk}\varphi + ZG\big(\partial_{ijk}\varphi - Z\partial_{ijk}\eta\big) - Z(\nabla\cdot v)\partial_{ijk}\eta \\ + \beta\nabla\cdot\big(A(\nabla\eta)\nabla\partial_{ijk}\eta\big) - \alpha\partial_{ijk}\eta + \mathcal{Q}_2^{ijk}[\eta,\varphi] + \mathcal{R}_2^{ijk}[\eta,\varphi], \end{aligned}
$$

where

$$
(7.64)\qquad \mathcal{Q}_2^{ijk}[\eta,\varphi] = \sum_\sigma \beta\,\nabla\cdot\Big(DA(\nabla\eta)\cdot(\partial_{\sigma(i)}\nabla\eta, \partial_{\sigma(j)\sigma(k)}\nabla\eta)\Big)
$$

the sum being taken on the circular permutations of $\{i,j,k\}$ and

$$
(7.65)\qquad \begin{aligned} \mathcal{R}_2^{ijk}[\eta,\varphi] \;=\; & -[\partial_{ij}, v]\cdot\nabla\partial_k\varphi + [\partial_{ij}, ZG]\partial_k\varphi + [\partial_{ij}, ZD_\eta G\varphi]\cdot\partial_k\eta \\ & +[\partial_{ij}, Zv]\cdot\nabla\partial_k\eta + \beta\nabla\cdot\big(D^2A(\nabla\eta)\cdot(\nabla\partial_k\eta, \nabla\partial_j\eta, \nabla\partial_i\eta)\big). \end{aligned}
$$

Again, as we shall see below, $\mathcal{R}_2^{ijk}$ can be considered as a semi-linear term while $\mathcal{Q}_2^{ijk}$ must be handled with care.

Now, let us set $U_{ijk} = (\partial_{ijk}\eta, \partial_{ijk}\varphi)^t$, then (7.58), (7.63) can be written under the abstract form

$$
(7.66)\qquad \partial_t U_{ijk} = J\big(\Lambda[\eta,\varphi]U_{ijk} + \mathcal{Q}^{ijk}[\eta,\varphi]\big) + \mathcal{R}^{ijk}[\eta,\varphi]
$$

where $\mathcal{R}^{ijk}[\eta,\varphi] = (\mathcal{R}_1^{ijk}[\eta,\varphi], \mathcal{R}_2^{ijk}[\eta,\varphi])^t$, $\mathcal{Q}^{ijk}[\eta,\varphi] = (-\mathcal{Q}_2^{ijk}[\eta,\varphi], \mathcal{Q}_1^{ijk}[\eta,\varphi])^t$ and $\Lambda[\eta,\varphi]$ is the linearized about (η,φ) operator. Namely

$$
\Lambda[\eta,\varphi] = \begin{pmatrix} -\beta\nabla\cdot(A(\nabla\eta)\nabla\,\cdot\,) + \alpha + ZG\big(Z\,\cdot\,\big) + Z\nabla\cdot v & v\cdot\nabla - \partial_x - ZG \\ -\nabla\cdot(v\,\cdot\,) - G\big(Z\,\cdot\,\big) + \partial_x & G \end{pmatrix},
$$

where $Z = Z[\eta,\varphi]$, $G = G[\eta]$, $v = v[\eta,\varphi]$.

By using Proposition 7.3 (with $\eta^a = \varphi^a = 0$ for the moment), a lengthy but straightforward computation shows that for k sufficiently large, we have the estimate

$$
\|\langle\partial\rangle^k\mathcal{R}_1^{ijk}\|_{H^1} + \|\langle\partial\rangle^k\mathcal{R}_2^{ijk}\|_{H^{\frac12}} \le \omega\big(\|\langle\partial\rangle^{k+3}\eta\|_{H^1} + \|\langle\partial\rangle^{k+3}\varphi\|_{H^{\frac12}}\big)\big(\|\langle\partial\rangle^{k+3}\eta\|_{H^1} + \|\langle\partial\rangle^{k+3}\varphi\|_{H^{\frac12}}\big)
$$

which indicates that we can consider $\mathcal{R}^{ijk}$ as a semi-linear term. We shall not prove this estimate now since we need to work in a more general framework where U is taken as a perturbation of V^a which is not in H^s. In the next section we describe the structure of this problem. The more general estimate that we shall prove below implies the above claimed estimate just by taking $V^a = 0$ in the following estimates.

7.4.2. *Analysis of* (7.2). Recall that $V^a = (\eta^a,\varphi^a) = Q + \delta U^a$, where U^a is the approximate solution given by Proposition 6.3. Coming back to the analysis of (7.2), we have that thanks to Proposition 6.3, with the notations introduced in the previous section, $U_{ijk} = (\partial_{ijk}\eta, \partial_{ijk}\varphi)^t$ solves

$$
(7.67)\qquad \partial_t U_{ijk} = J\Big(\Lambda^\delta U_{ijk} + (\mathcal{Q}^{ijk})^\delta - (\mathcal{Q}^{ijk})^a\Big) + \mathcal{G}^{ijk}[\eta,\varphi] - \partial_{ijk}R^{ap}, \quad U(0) = 0,
$$

where the claimed semi-linear term is now given by

$$
(7.68)\qquad \mathcal{G}^{ijk}[\eta,\varphi] = \mathcal{G}_1^{ijk}[\eta,\varphi] + \mathcal{G}_2^{ijk}[\eta,\varphi],
$$

where

$$
\mathcal{G}_1^{ijk}[\eta,\varphi] = \mathcal{R}^{ijk}[\eta+\eta^a, \varphi+\varphi^a] - \mathcal{R}^{ijk}[\eta^a,\varphi^a] = (\mathcal{R}^{ijk})^\delta - (\mathcal{R}^{ijk})^a
$$

and

$$
\mathcal{G}_2^{ijk}[\eta,\varphi] = J\big(\Lambda[\eta+\eta^a,\varphi+\varphi^a] - \Lambda[\eta^a,\varphi^a]\big)\partial_{ijk}V^a = J\big(\Lambda^\delta - \Lambda^a\big)\partial_{ijk}V^a\,.
$$

The terms $\partial_{ijk}R^{ap}$ enjoys the bounds provided by Proposition 6.3. It will also be useful to use the shorter notation

$$
(7.69)\qquad \partial_t U_{ijk} = J(\Lambda^\delta U_{ijk} + (\mathcal{Q}^{ijk})^\delta - (\mathcal{Q}^{ijk})^a) + F_{ijk}\,,
$$

for (7.67) where

$$F_{ijk} = \mathcal{G}^{ijk}[\eta,\varphi] - \partial_{ijk}R^{ap}\,. \tag{7.70}$$

In order to perform our energy estimates, we shall use the canonical form of (7.69) identified in [24] (in the absence of surface tension). We set $W_{ijk} = PU_{ijk}$, where

$$P \equiv \begin{pmatrix} 1 & 0 \\ -Z[U+V^a] & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -Z^\delta & 1 \end{pmatrix}.$$

We find that W_{ijk} solves the problem

$$\partial_t W_{ijk} = J\big(L^\delta W_{ijk} - JPJ\big((\mathcal{Q}^{ijk})^\delta - (\mathcal{Q}^{ijk})^a\big)\big) + PF_{ijk}, \tag{7.71}$$

where

$$L[\eta,\varphi] = \begin{pmatrix} -\mathcal{P}(\nabla\eta) + (v\cdot\nabla Z) + (\partial_t - \partial_x)Z & v\cdot\nabla - \partial_x \\ -\nabla\cdot(v\cdot) + \partial_x & G \end{pmatrix},$$

with $Z = Z[\eta,\varphi]$, $v = v[\eta,\varphi]$, $G = G[\eta]$ and $\mathcal{P}(\nabla\eta)$ defined by $\mathcal{P}(\nabla\eta) \equiv \beta\nabla\cdot(A(\nabla\eta)\nabla\cdot) - \alpha$.

7.5. **Estimates on the semi-linear terms.** In this section we estimate the semilinear term $\mathcal{G}^{ijk}[\eta,\varphi]$ arising in (7.67). In the estimate below V^a will be evaluated in $\mathcal{W}^{m+S}$ with S sufficiently large since we will make use of Propositions 7.3 and 7.13. Here is the main result of this section.

Proposition 7.17. *For $m \geq 2$, and $S \geq 5$, we have the estimate*

$$\|\mathcal{G}^{ijk}[\eta,\varphi]\|_{X^m} \leq \omega\big(\|V^a\|_{\mathcal{W}^{m+S}} + \|U\|_{X^{m+3}}\big)\|U\|_{X^{m+3}}.$$

Since there are many terms that we need to estimate, we shall split the proof of Proposition 7.17 in many Propositions and Lemmas. To save place, we shall use the notation

$$\overline{\omega}_{m,S} = \omega\big(\|V^a\|_{\mathcal{W}^{m+S}} + \|U\|_{X^{m+3}}\big).$$

Our first result towards the proof of Proposition 7.17 is:

Proposition 7.18. *For $m \geq 2$ and $S \geq 5$, we have*

$$\big\|\langle\partial\rangle^m\big((\mathcal{R}_1^{ijk})^\delta - (\mathcal{R}_1^{ijk})^a\big)\big\|_{H^1} \leq \overline{\omega}_{m,S}\|U\|_{X^{m+3}}. \tag{7.72}$$

Proof. From the definition of $\mathcal{R}_1^{ijk}$, we have to estimate the H^1 norm of terms like

$$\langle\partial\rangle^m\Big(D_\eta^n G^\delta\partial^\gamma(\varphi^a+\varphi)\cdot\big(\partial^{\beta_1}(\eta^a+\eta),\cdots,\partial^{\beta_n}(\eta^a+\eta)\big) - D_\eta^n G^a\partial^\gamma\varphi^a\cdot\big(\partial^{\beta_1}\eta^a,\cdots,\partial^{\beta_n}\eta^a\big)\Big)$$

where n, γ and β_i satisfies the constraints (7.61). By multilinearity and symmetry, we need to estimate three types of terms:

$$I_1 = D_\eta^n G^\delta\partial^\gamma\varphi^a\cdot\big(\partial^{\beta_1}\eta^a,\cdots,\partial^{\beta_n}\eta^a\big) - D_\eta^n G^a\partial^\gamma\varphi^a\cdot\big(\partial^{\beta_1}\eta^a,\cdots,\partial^{\beta_n}\eta^a\big),$$

$$I_2 = D_\eta^n G^\delta\partial^\gamma\varphi\cdot\big(\partial^{\overline{\beta}_1}\eta^a,\cdots,\partial^{\overline{\beta}_l}\eta^a,\partial^{\overline{\beta}_{l+1}}\eta,\cdots,\partial^{\overline{\beta}_n}\eta\big)$$

and

$$I_3 = D_\eta^n G^\delta\partial^\gamma\varphi^a\cdot\big(\partial^{\overline{\beta}_1}\eta^a,\cdots,\partial^{\overline{\beta}_l}\eta^a,\partial^{\overline{\beta}_{l+1}}\eta,\cdots,\partial^{\overline{\beta}_n}\eta\big)$$

with $0 \leq l \leq n$ for I_2, $0 \leq l \leq n-1$ for I_3 and $\overline{\beta}_i = \beta_{\sigma(i)}$ for some permutation σ of $\{1\cdots,n\}$. In particular, from (7.61), we have that $|\overline{\beta}_i| \leq 2$ and $|\gamma| \leq 1$. From the second estimate of Proposition 7.3, we immediately get that $\|\langle\partial\rangle^m I_2\|_{H^1}$ satisfies the claimed estimate (7.72) that is

$$\|\langle\partial\rangle^m I_2\|_{H^1} \leq \overline{\omega}_{m,S}\|U\|_{X^{m+3}}.$$

In a similar way, the estimate

$$\|\langle\partial\rangle^m I_3\|_{H^1} \leq \overline{\omega}_{m,S}\|U\|_{X^{m+3}}$$

follows from Proposition 7.13 estimate (7.46).

For the estimate of I_1, we first write

$$
\begin{aligned}
(7.73)\qquad I_1 &= \int_0^1 \frac{d}{ds}\Big(D_\eta^n G[\eta^a+s\eta]\partial^\gamma\varphi^a\cdot\big(\partial^{\beta_1}\eta^a,\cdots,\partial^{\beta_n}\eta^a\big)\Big)ds\\
&= \int_0^1 D_\eta^{n+1}G[\eta^a+s\eta]\partial^\gamma\varphi^a\cdot\big(\partial^{\beta_1}\eta^a,\cdots,\partial^{\beta_n}\eta^a,\eta\big)ds\,.
\end{aligned}
$$

From Proposition 7.13 estimate (7.46) (with η being the only function in H^s) and (7.61), we thus also obtain that

$$\|\langle\partial\rangle^m I_1\|_{H^1}\le\overline{\omega}_{m,S}\|\langle\partial\rangle^{m+1}\eta\|_{H^1}.$$

This ends the proof of Proposition 7.18. □

Remark 7.19. *Note that the important point in the proof of Proposition 7.18 is that η does not appear with more than two derivatives in the directions of the Frechet derivatives of G and that φ does not appear with more than two derivatives, i.e. we are in the scope of applicability of Propositions 7.3 and Proposition 7.13.*

We shall now turn to the estimate of the terms involving $\mathcal{R}_2^{ijk}$. Towards this, we shall use very often the following product estimates:

Proposition 7.20. *For $m\ge 2$, $\sigma=1/2$ or $\sigma=1$, we have*

$$
\begin{aligned}
(7.74)\qquad \|\langle\partial\rangle^m(uv)\|_{H^\sigma(\mathbb{R}^2)} &\le C\|\langle\partial\rangle^m u\|_{H^\sigma(\mathbb{R}^2)}\|\langle\partial\rangle^m v\|_{H^\sigma(\mathbb{R}^2)}\,,\\
(7.75)\qquad \|\langle\partial\rangle^m(uv)\|_{H^\sigma(\mathbb{R}^2)} &\le C\|\langle\partial\rangle^m u\|_{W^{1,\infty}(\mathbb{R}^2)}\|\langle\partial\rangle^m v\|_{H^\sigma(\mathbb{R}^2)}\,.
\end{aligned}
$$

Note that in the second estimate u and v do not play symmetric parts. As in Lemma 7.12, a sharper but not needed estimate holds by replacing $\|\langle\partial\rangle^m u\|_{W^{1,\infty}(\mathbb{R}^2)}$ with $\|\langle\partial\rangle^m u\|_{\mathcal{C}^\alpha}$ for $\alpha>1/2$ when $\sigma=1/2$.

The proof of Proposition 7.20 is very similar to the ones of Lemma 7.2 and Lemma 7.12 and hence will be omitted. In the next proposition, we evaluate the contribution of $\mathcal{R}_2^{ijk}$.

Proposition 7.21. *For $m\ge 2$, $S\ge 5$, we have the estimate*

$$\big\|\langle\partial\rangle^m\big((\mathcal{R}_2^{ijk})^\delta-(\mathcal{R}_2^{ijk})^a\big)\big\|_{H^{\frac12}}\le\overline{\omega}_{m,S}\|U\|_{X^{m+3}}$$

Proof of Proposition 7.21. We shall first establish the following lemma.

Lemma 7.22. *For $m\ge 2$, $S\ge 5$, we have*

$$(7.76)\qquad \|\langle\partial\rangle^{m+2}(v^\delta-v^a)\|_{H^{\frac12}}+\|\langle\partial\rangle^{m+2}(Z^\delta-Z^a)\|_{H^{\frac12}}\le\overline{\omega}_{m,S}\|U\|_{X^{m+3}}$$

and for $|\gamma|\le 2$, we also have

$$(7.77)\qquad \big\|\langle\partial\rangle^m\big(\partial^\gamma v^\delta\,\psi\big)\big\|_{H^{\frac12}}+\big\|\langle\partial\rangle^m\big(\partial^\gamma Z^\delta\,\psi\big)\big\|_{H^{\frac12}}\le\overline{\omega}_{m,S}\|\langle\partial\rangle^m\psi\|_{H^{\frac12}}.$$

Proof. We start with the estimates involving Z. Write $Z=Z_1+Z_2$, where $Z_1=(1+|\nabla\eta|^2)^{-1}G[\eta]\varphi$. A straightforward application of Proposition 7.20 implies that

$$(7.78)\qquad \|\langle\partial\rangle^{m+2}(Z_2^\delta-Z_2^a)\|_{H^{\frac12}}\le\overline{\omega}_{m,S}\|U\|_{X^{m+3}}\,.$$

We shall not detail the proof of the inequality (7.78). Instead we shall estimate in detail the contribution of Z_1 whose proof is very similar and contains additional difficulties coming from the presence of the Dirichlet-Neumann operator. Set $w[\eta]\equiv(1+|\nabla\eta|^2)^{-1}$. Then we can write

$$(7.79)\qquad Z_1^\delta-Z_1^a = (w^\delta-w^a)G^a\varphi^a+w^\delta(G^\delta-G^a)\varphi^a+w^\delta G^\delta\varphi\,.$$

To estimate the first term in the decomposition (7.79), we use Proposition 7.20 to obtain

$$(7.80)\qquad \|\langle\partial\rangle^{m+2}\big((w^\delta-w^a)G^a\varphi^a\big)\|_{H^{\frac12}}\le C\|\langle\partial\rangle^{m+2}(w^\delta-w^a)\|_{H^{\frac12}}\|\langle\partial\rangle^{m+2}(G^a\varphi^a)\|_{W^{1,\infty}}\,.$$

Next, since

$$w^\delta - w^a = -\frac{|\nabla\eta + \nabla\eta^a|^2 - |\nabla\eta^a|^2}{(1+|\nabla\eta^a|^2)(1+|\nabla\eta+\nabla\eta^a|^2)}$$

we obtain after several applications of Proposition 7.20 that:

$$\|\langle\partial\rangle^{m+2}(w^\delta - w^a)\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\|U\|_{X^{m+3}}\,. \tag{7.81}$$

Since, by using the estimate (7.44) of Proposition 7.13, we also have that for $S \ge 5$

$$\|\langle\partial\rangle^{m+2}(G[\eta^a]\varphi^a)\|_{W^{1,\infty}} \le \omega\big(\|V^a\|_{\mathcal{W}^{m+S}}\big)$$

we get that

$$\|\langle\partial\rangle^{m+2}\big((w^\delta - w^a)G^a\varphi^a\big)\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\|U\|_{X^{m+3}}\,. \tag{7.82}$$

Next, to estimate the second term in (7.79), we first observe that for every ψ smooth enough, the bound

$$\|\langle\partial\rangle^{m+2}\big(w^\delta\psi\big)\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\|\langle\partial\rangle^{m+2}\psi\|_{H^{\frac12}} \tag{7.83}$$

holds. Indeed, it suffices to write $w^\delta = (w^\delta - w^a) + w^a$ and to use Proposition 7.20 and (7.81). By using (7.83), we thus obtain

$$\|\langle\partial\rangle^{m+2}(w^\delta(G^\delta - G^a)\varphi^a)\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\|\langle\partial\rangle^{m+2}((G^\delta - G^a)\varphi^a)\|_{H^{\frac12}}\,. \tag{7.84}$$

To estimate the last above term, we use that for $|\alpha| = 2$

$$\partial^\alpha\big((G^\delta - G^a)\varphi^a\big) = \partial^\alpha\Big(\int_0^1 D_\eta G[s\eta + \eta^a]\varphi^a\cdot\eta ds\Big)$$

and we notice that $\partial^\alpha\big(D_\eta G[s\eta+\eta^a]\varphi^a\cdot\eta\big)$ can be expanded as a sum of terms under the form

$$\int_0^1 D^n_\eta G[\eta^a + s\eta]\partial^\beta\varphi^a\cdot\big(\partial^{\gamma_1}\eta, \partial^{\gamma_2}h_1, \cdots, \partial^{\gamma_n}h_{n-1}\big)\,ds$$

where $n \ge 1$, the h_i may be η or η^a and $|\gamma_i| \le 2$. Since $\partial^{\gamma_1}\eta$ belongs to the Sobolev scale and there is never more than two derivatives of η involved in the above expression, we can again use the estimate (7.46) of Proposition 7.13 to infer that

$$\|\langle\partial\rangle^{m+2}((G^\delta - G^a)\varphi^a)\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\|U\|_{X^{m+3}}\,.$$

Coming back to (7.84), we have thus proven that

$$\|\langle\partial\rangle^{m+2}(w^\delta(G^\delta - G^a)\varphi^a)\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\|U\|_{X^{m+3}}\,. \tag{7.85}$$

Finally, to estimate the last term in (7.79), we first write from another use of (7.83) that

$$\|\langle\partial\rangle^{m+2}(w^\delta G^\delta\varphi)\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\|\langle\partial\rangle^{m+2}(G^\delta\varphi)\|_{H^{\frac12}}\,.$$

Since for $|\alpha| = 2$, we can decompose $\partial^\alpha\big(G[\eta+\eta^a]\varphi\big)$ as a sum of terms of the form

$$D^n_\eta G[\eta+\eta^a]\partial^\gamma\varphi\cdot(h_1, \cdots h_n)$$

with $h_i = \partial^{\beta_i}\eta$ or $\partial^{\beta_i}\eta^a$ and $|\gamma| \le 2$, $|\beta_i| \le 2$, we can use Proposition 7.3 to obtain

$$\|\langle\partial\rangle^{m+2}(G^\delta\varphi)\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\|\langle\partial\rangle^{m+3}\varphi\|_{H^{\frac12}}$$

and therefore, we obtain that

$$\|\langle\partial\rangle^{m+2}(w^\delta G^\delta\varphi)\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\|U\|_{X^{m+3}}. \tag{7.86}$$

By using the decomposition (7.79) and the estimates (7.82), (7.85), (7.86), we find that

$$\|\langle\partial\rangle^{m+2}(Z_1^\delta - Z_1^a)\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\,\|U\|_{X^{m+3}}$$

which in turn taking into account (7.78) gives

$$\|\langle\partial\rangle^{m+2}(Z^\delta - Z^a)\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\,\|U\|_{X^{m+3}}\,. \tag{7.87}$$

This proves the first estimate for Z in Lemma 7.22. Next, since can write for $|\gamma|\le 2$ that

$$\|\langle\partial\rangle^m(\partial^\gamma Z^\delta\,\psi)\|_{H^{\frac12}} \le \|\langle\partial\rangle^m\big(\partial^\gamma(Z^\delta - Z^a)\psi\big)\|_{H^{\frac12}} + \|\langle\partial\rangle^m(\partial^\gamma Z^a\,\psi)\|_{H^{\frac12}},$$

by using (7.87) together with Proposition 7.20 and Proposition 7.13 (to bound Z^a) we also find

$$\|\langle\partial\rangle^m(\partial^\gamma Z^\delta\,\psi)\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\,\|\langle\partial\rangle^m\psi\|_{H^{\frac12}} \tag{7.88}$$

and hence the second estimate for Z is proven.

It remains to prove the claimed estimates for v. Let us recall that $v[\eta,\varphi] = \nabla\varphi - Z[\eta,\varphi]\nabla\eta$. Therefore we can write

$$v^\delta - v^a = \nabla\varphi - \Big((Z^\delta - Z^a)\nabla\eta^a + Z^\delta\nabla\eta\Big).$$

Consequently, by using (7.87), (7.88) and Lemma 7.20, we infer

$$\|\langle\partial\rangle^{m+2}(v^\delta - v^a)\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\,\|U\|_{X^{m+3}}. \tag{7.89}$$

Finally, we can write that

$$\|\langle\partial\rangle^m(\partial^\gamma\, v^\delta\psi)\|_{H^{\frac12}} \le \|\langle\partial\rangle^m(\partial^\gamma(v^\delta - v^a)\psi)\|_{H^{\frac12}} + \|\langle\partial\rangle^m(\partial^\gamma v^a\psi)\|_{H^{\frac12}}$$

and hence, by using (7.89) together with Proposition 7.20 and Proposition 7.13 (to bound v^a) we get the bound

$$\|\langle\partial\rangle^m(\partial^\gamma v^\delta\,\psi)\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\,\|\langle\partial\rangle^m\psi\|_{H^{\frac12}}\,. \tag{7.90}$$

This ends the proof of Lemma 7.22. □

The result of Lemma 7.22 will be one of the main tool when estimating $\mathcal{R}_2^{ijk}[V^a+U] - \mathcal{R}_2^{ijk}[V^a]$. We start by the estimate of the contribution of the following terms in the definition (7.65) of $\mathcal{R}_2^{ijk}$.

Lemma 7.23. *For $S\ge 5$, $m\ge 2$, we have the following estimates:*

$$\big\|\langle\partial\rangle^m\big([\partial_{ij}, v^\delta]\cdot\partial_k\nabla\varphi^\delta - [\partial_{ij}, v^a]\cdot\partial_k\nabla\varphi^a\big)\big\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\,\|U\|_{X^{m+3}}, \tag{7.91}$$

$$\big\|\langle\partial\rangle^m\big([\partial_{ij}, Z^\delta G^\delta](\partial_k\varphi^\delta) - [\partial_{ij}, Z^a G^a](\partial_k\varphi^a)\big)\big\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\,\|U\|_{X^{m+3}}, \tag{7.92}$$

$$\big\|\langle\partial\rangle^m\big([\partial_{ij}, Z^\delta v^\delta]\cdot\nabla\partial_k\eta^\delta - [\partial_{ij}, Z^a v^a]\cdot\nabla\partial_k\eta^a\big)\big\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\|U\|_{X^{m+3}}, \tag{7.93}$$

$$\Big\|\langle\partial\rangle^m\big(\big[\partial_{ij}, Z^\delta D_\eta G^\delta\varphi^\delta\big]\cdot\partial_k\eta^\delta - \big[\partial_{ij}, Z^a D_\eta G^a\varphi^a\big]\cdot\partial_k\eta^a\big)\Big\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\|U\|_{X^{m+3}}, \tag{7.94}$$

where we again use the notation $\overline{\omega}_{m,S} = \omega\big(\|V^a\|_{\mathcal{W}^{m+S}} + \|U\|_{X^{m+3}}\big)$.

Note that v and Z both satisfy the estimates of Lemma 7.22 and that the Dirichlet-Neumann operator G acts as a first order operator in space. Consequently, the statements (7.92) and (7.91) are formally very close. Since the Dirichlet-Neumann operator which has a nonlinear dependence in the surface is much more difficult to deal with than the simple operator ∇, we shall only focus on the proof of (7.92). The proof of (7.91) which is simpler and follows the same lines is left to the reader. In the same way, since because of Lemma 7.22, Zv behaves as v and Z, the proof of (7.93) which is very similar and simpler than the one of (7.92) is left to the reader. Note that the proof of (7.93) is even simpler since η is smoother than φ: we are allowed to put it in H^1.

Proof of Lemma 7.23. As already explained, we focus on the proof of (7.92) and (7.94). Let us start with the proof of (7.92). The commutator $[\partial_{ij}, ZG]\partial_k\varphi$ can be expanded as a sum of terms under the form

$$I[U] = \partial^{\gamma_0} Z\, D^n_\eta G[\eta]\partial_k\partial^\beta\varphi \cdot \big(\partial^{\gamma_1}\eta, \cdots, \partial^{\gamma_n}\eta\big) \tag{7.95}$$

where the indices satisfy the constraints

$$\forall i,\ 0 \le i \le n,\ |\gamma_i| \le 2, \quad |\beta| \le 1. \tag{7.96}$$

To get (7.92), we thus need to estimate $\|\langle\partial\rangle^m\big(I^\delta - I^a\big)\|_{H^{\frac12}}$. Towards this, we expand

$$\begin{aligned}
I^\delta - I^a \ &= \ \big(\partial^{\gamma_0} Z^\delta - \partial^{\gamma_0} Z^a\big) D^n_\eta G^a \partial_k\partial^\beta\varphi^a \cdot \big(\partial^{\gamma_1}\eta^a, \cdots, \partial^{\gamma_n}\eta^a\big)\\
&\quad + \partial^{\gamma_0} Z^\delta \big(D^n_\eta G^\delta - D^n_\eta G^a\big)\partial_k\partial^\beta\varphi^a \cdot \big(\partial^{\gamma_1}\eta^a, \cdots, \partial^{\gamma_n}\eta^a\big)\\
&\quad + \partial^{\gamma_0} Z^\delta D^n_\eta G^\delta \partial_k\partial^\beta\varphi \cdot \big(\partial^{\gamma_1}\eta^a, \cdots, \partial^{\gamma_n}\eta^a\big)\\
&\quad + \mathcal{J}\\
&\equiv \ I_1 + I_2 + I_3 + \mathcal{J} \qquad (7.97)
\end{aligned}$$

where $\mathcal{J}$ is a sum of terms under the form

$$\mathcal{J}_l = \partial^{\gamma_0} Z^\delta D^n_\eta G^\delta \partial_k\varphi^\delta \cdot \big(\partial^{\overline{\gamma}_1}\eta^a, \cdots, \partial^{\overline{\gamma}_l}\eta^a, \partial^{\overline{\gamma}_{l+1}}\eta, \cdots \partial^{\overline{\gamma}_n}\eta\big),$$

where the $\overline{\gamma}_i$ are obtained from the γ_i by a permutation and l is such that $l \le n-1$.

By using (7.75), (7.45), (7.76) and the fact that $|\gamma_0| \le 2$, we obtain

$$\begin{aligned}
\|\langle\partial\rangle^m I_1\|_{H^{\frac12}} \ &\le \ C\|\langle\partial\rangle^m D^n_\eta G^a \partial_k\partial^\beta\varphi^a \cdot \big(\partial^{\gamma_1}\eta^a, \cdots, \partial^{\gamma_n}\eta^a\big)\|_{W^{1,\infty}} \|\langle\partial\rangle^{m+2}\big(Z^\delta - Z^a\big)\|_{H^{\frac12}}\\
&\le \overline{\omega}_{m,S}\|U\|_{X^{m+3}}.
\end{aligned}$$

Next, since we have from (7.77) that

$$\|\langle\partial\rangle^m\big(\partial^{\gamma_0} Z^\delta D^n_\eta G^\delta \partial_k\partial^\beta\varphi\cdot\big(\partial^{\gamma_1}\eta^a, \cdots, \partial^{\gamma_n}\eta^a\big)\big)\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\|\langle\partial\rangle^m\big(D^n_\eta G^\delta \partial_k\partial^\beta\varphi\cdot\big(\partial^{\gamma_1}\eta^a, \cdots, \partial^{\gamma_n}\eta^a\big)\big)\|_{H^{\frac12}},$$

we get by using (7.8) with $l = n$ that

$$\|\langle\partial\rangle^m\big(D^n_\eta G^\delta \partial_k\partial^\beta\varphi \cdot \big(\partial^{\gamma_1}\eta^a, \cdots, \partial^{\gamma_n}\eta^a\big)\big)\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\|\langle\partial\rangle^{m+1}\partial_k\partial^\beta\varphi\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\|\langle\partial\rangle^{m+3}\varphi\|_{H^{\frac12}}.$$

This yields by using (7.96) that

$$\|\langle\partial\rangle^m I_3\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\|\langle\partial\rangle^{m+3}\varphi\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\|U\|_{X^{m+3}}.$$

In a similar way, we can use (7.77) to get

$$\|\langle\partial\rangle^m I_2\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\|\langle\partial\rangle^m\big(\big(D^n_\eta G^\delta - D^n_\eta G^a\big)\partial_k\partial^\beta\varphi^a \cdot \big(\partial^{\gamma_1}\eta^a, \cdots, \partial^{\gamma_n}\eta^a\big)\big)\|_{H^1}.$$

Therefore, by a new use of the Taylor formula as in (7.73) and (7.46) we infer

$$\|\langle\partial\rangle^m I_2\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\|\langle\partial\rangle^{m+1}\eta\|_{H^1} \le \overline{\omega}_{m,S}\|U\|_{X^{m+3}}.$$

Finally, to estimate $\mathcal{J}_l$, since $\varphi^\delta = \varphi + \varphi^a$, we can also use successively (7.77) and (7.8) for the part involving φ or (7.46) since $l \le n-1$ for the part involving φ^a to get

$$\|\langle\partial\rangle^m \mathcal{J}_l\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\|U\|_{X^{m+3}}.$$

Consequently, summarizing the previous estimates and (7.97), we have proven that

$$\|\langle\partial\rangle^m\big(I^\delta - I^a\big)\|_{H^{\frac12}} \le \overline{\omega}_{m,S}\|U\|_{X^{m+3}}. \tag{7.98}$$

This ends the proof of (7.92).

Let us turn to the proof of (7.94). The commutator $\big[\partial_{ij}, ZD_\eta G\varphi\big]\cdot\partial_k\eta$ can be expanded in a sum of terms under the form

$$\partial^{\gamma_0} ZD_\eta^n G[\eta]\partial^\beta\varphi\cdot\big(\partial^{\gamma_1}\eta,\cdots,\partial^{\gamma_{n-1}}\eta,\partial^\gamma\partial_k\eta\big)$$

where the indices satisfy the constraints

$$n\geq 1,\quad |\gamma_0|+|\beta|+|\gamma_1|+\cdots+|\gamma_{n-1}|+|\gamma|=2,\quad |\gamma|\leq 1.$$

Consequently, in view of (7.95), (7.96), we see that the number of derivatives on φ is the same as in (7.95) and that $1+|\gamma|\leq 2$. Consequently, (7.94) also follows from (7.98). This ends the proof of Lemma 7.23. □

To prove Proposition 7.21, we shall also need the following statement.

Lemma 7.24. *For $m\geq 2$ and $S\geq 5$,*

$$\big\|\langle\partial\rangle^m\big(\nabla\cdot\big(D^2A(\nabla\eta^\delta)\cdot\big(\nabla\partial_k\eta^\delta,\nabla\partial_j\eta^\delta,\nabla\partial_i\eta^\delta\big)\big)-\nabla\cdot\big(D^2A(\nabla\eta^a)\cdot(\nabla\partial_k\eta^a,\nabla\partial_j\eta^a,\nabla\partial_i\eta^a)\big)\big)\big\|_{H^{\frac12}}$$
$$\leq\overline{\omega}_{m,S}\|U\|_{X^{m+3}}.$$

Proof. We first expand

$$D^2A(\nabla(\eta+\eta^a))\cdot(\nabla\partial_k(\eta+\eta^a),\nabla\partial_j(\eta+\eta^a),\nabla\partial_i(\eta+\eta^a)\big)-D^2A(\nabla\eta^a)\cdot(\nabla\partial_k\eta^a,\nabla\partial_j\eta^a,\nabla\partial_i\eta^a\big)$$

as

$$\begin{aligned}
&(D^2A(\nabla(\eta+\eta^a))-D^2A(\nabla\eta^a))\cdot(\nabla\partial_k\eta^a,\nabla\partial_j\eta^a,\nabla\partial_i\eta^a\big)\\
+\ &D^2A(\nabla(\eta+\eta^a))\cdot(\nabla\partial_k\eta,\nabla\partial_j\eta^a,\nabla\partial_i\eta^a\big)\\
+\ &D^2A(\nabla(\eta+\eta^a))\cdot(\nabla\partial_k(\eta+\eta^a),\nabla\partial_j\eta,\nabla\partial_i\eta^a\big)\\
+\ &D^2A(\nabla(\eta+\eta^a))\cdot(\nabla\partial_k(\eta+\eta^a),\nabla\partial_j(\eta+\eta^a),\nabla\partial_i\eta\big)
\end{aligned}$$

and we rewrite the first term as

$$\begin{aligned}
&(D^2A(\nabla(\eta+\eta^a))-D^2A(\nabla\eta^a))\cdot(\nabla\partial_k\eta^a,\nabla\partial_j\eta^a,\nabla\partial_i\eta^a\big)\\
&=\int_0^1 D^3A(\nabla(s\eta+\eta^a))\cdot(\eta,\nabla\partial_k\eta^a,\nabla\partial_j\eta^a,\nabla\partial_i\eta^a\big)ds.
\end{aligned}$$

Let us observe that in all the above terms, we have at most two derivatives of η involved. We can then apply ∇ to these terms which implies that in the end we have at most three derivative of η involved. Since we need an $H^{\frac12}$ estimate after applying $\langle\partial\rangle^m$, we can complete the proof of Lemma 7.24 by coming back to the definition of A and using the classical Moser type estimates. □

End of the proof of Proposition 7.21. The combination of Lemma 7.23 and Lemma 7.24 completes the proof of Proposition 7.21.

Note that Proposition 7.18 and Proposition 7.21 lead to the following statement.

Corollary 7.25. *For $m\geq 2$ and $S\geq 5$, we have*

$$\|\mathcal{G}_1^{ijk}[\eta,\varphi]\|_{X^m}\leq\omega\big(\|V^a\|_{\mathcal{W}^{m+S}}+\|U\|_{X^{m+3}}\big)\|U\|_{X^{m+3}}\,.$$

To end the proof of Proposition 7.17, it remains to estimate $\mathcal{G}_2^{ijk}$. We have the following statement.

Proposition 7.26. *For $m\geq 2$ and $S\geq 5$, we have the estimate*

$$\|\mathcal{G}_2^{ijk}[\eta,\varphi]\|_{X^m}\leq\overline{\omega}_{m,S}\|U\|_{X^{m+3}}\,.$$

Proof. We need to show that

$$\|\langle\partial\rangle^m(I[U+V^a]-I[V^a])\|_{H^1}\le\overline{\omega}_{m,S}\|U\|_{X^{m+3}} \tag{7.99}$$

and

$$\|\langle\partial\rangle^m(J[U+V^a]-J[V^a])\|_{H^{\frac12}}\le\overline{\omega}_{m,S}\|U\|_{X^{m+3}}, \tag{7.100}$$

where

$$I[\eta,\varphi]=-\nabla\cdot\big(v[\eta,\varphi]\partial_{ijk}\eta^a\big)-G[\eta]\big(Z[\eta,\varphi]\partial_{ijk}\eta^a\big)+G[\eta]\partial_{ijk}\varphi^a$$

and

$$\begin{aligned}J[\eta,\varphi] \;=\;& \beta\nabla\cdot\big(A(\nabla\eta)\nabla\partial_{ijk}\eta^a\big)-Z[\eta,\varphi]G[\eta]\big(Z[\eta,\varphi]\partial_{ijk}\eta^a\big)\\ &-Z[\eta,\varphi]\,\nabla\cdot v[\eta,\varphi]\,\partial_{ijk}\eta^a-v[\eta,\varphi]\cdot\nabla\partial_{ijk}\varphi^a+Z[\eta,\varphi]G[\eta]\partial_{ijk}\varphi^a\,.\end{aligned}$$

Again, we observe that to get (7.99) and (7.100), it suffices to use again Proposition 7.3, Proposition 7.13 and Lemma 7.22. All the terms that we have to handle are very similar to the ones that we have estimated in the proofs of Lemma 7.23 and Lemma 7.24, consequently, we shall not give more details. This ends the proof of Proposition 7.26. □

End of the proof of Proposition 7.17. It suffices to combine the statements of Corollary 7.25 and Proposition 7.26 in view of (7.68).

7.6. **Estimates of the subprincipal terms.** In this subsection, we turn to the estimates on the subprincipal term of (7.67) namely the term $(\mathcal{Q}^{ijk})^\delta-(\mathcal{Q}^{ijk})^a$.

Proposition 7.27. *For $m\ge2$ and $S\ge5$, we have*

$$\|\langle\partial\rangle^m\big((\mathcal{Q}_1^{ijk})^\delta-(\mathcal{Q}_1^{ijk})^a\big)\|_{H^{-\frac12}}\le\overline{\omega}_{m,S}\|U\|_{X^{m+2}}\,. \tag{7.101}$$

and

$$\|\langle\partial\rangle^m\big((\mathcal{Q}_2^{ijk})^\delta-(\mathcal{Q}_2^{ijk})^a\big)\|_{H^{-1}}\le\overline{\omega}_{m,S}\|U\|_{X^{m+2}}\,. \tag{7.102}$$

Proof. We first prove (7.101). Thanks to (7.59), we need to estimate $\|\langle\partial\rangle^m I_{ijk}\|_{H^{-\frac12}}$ with

$$I_{ijk}=D_\eta G^\delta\partial_{jk}\varphi^\delta\cdot\partial_i\eta^\delta-D_\eta G^a\partial_{jk}\varphi^a\cdot\partial_i\eta^a$$

(and similar expression by cycle change of i,j,k). We can expand this expression as

$$(D_\eta G^\delta-D_\eta G^a)\partial_{jk}\varphi^a\cdot\partial_i\eta^a+D_\eta G^\delta\partial_{jk}\varphi\cdot\partial_i\eta^a+D_\eta G^\delta\partial_{jk}\varphi^\delta\cdot\partial_i\eta\,.$$

Now we observe that each term in the above decomposition is in the scope of applicability of Proposition 7.3 or Proposition 7.13. Indeed the first term can be estimated by invoking Proposition 7.13 after writing

$$(D_\eta G^\delta-D_\eta G^a)\partial_{jk}\varphi^a\cdot\partial_i\eta^a=\int_0^1 D_\eta^2G[s\eta+\eta^a]\partial_{jk}\varphi^a\cdot(\partial_i\eta^a,\eta)ds.$$

Observe that in this place we can afford to crudely estimate the $H^{-\frac12}$ norm by the H^1 norm controlled by Proposition 7.13. The second term can be estimated in $H^{-\frac12}$ by invoking Proposition 7.3. The third term can be written as

$$D_\eta G^\delta\partial_{jk}\varphi\cdot\partial_i\eta+D_\eta G^\delta\partial_{jk}\varphi^a\cdot\partial_i\eta. \tag{7.103}$$

Then the first term in the decomposition (7.103) can be estimated in $H^{-\frac12}$ by using Proposition 7.3 while the second term is in the scope of applicability of Proposition 7.13, again after crudely estimating the $H^{-\frac12}$ norm by the H^1 norm. This ends the proof of (7.101).

Let us turn to the proof of (7.102). By using (7.64), a duality argument and an integration by parts, we obtain that the issue is to evaluate

$$\|\langle\partial\rangle^m(DA(\nabla(\eta^\delta)\cdot(\partial_i\nabla\eta^\delta,\partial_{jk}\nabla\eta^\delta)-DA(\nabla(\eta^a))\cdot(\partial_i\nabla\eta^a,\partial_{jk}\nabla\eta^a)))\|_{L^2} \tag{7.104}$$

(and similar expression by cycle change of i, j, k). For that purpose, it suffices to observe that in the definition of the expression (7.104) there are at most $m+3$ derivatives of η involved and that at least one of them is a spatial derivative. Then the estimate of the L^2 norm of (7.104) follows from the Moser type estimates of Proposition 7.20 after some decompositions in the spirit of the proof of Lemma 7.24. This completes the proof of Proposition 7.27. □

7.7. **The quadratic form associated to L^δ.** The last step before turning to the proof of the energy estimates is the study of the quadratic form associated to the operator $L^\delta = L[U^\delta]$ which arises as the main part of (7.71).

Proposition 7.28. *We have the estimates*

$$(L^\delta W, W) + \|W\|_{L^2}^2 \geq \frac{1}{\overline{\omega}_{2,5}}\|W\|_{X^0}^2 \tag{7.105}$$

$$|(L^\delta W, V)| \leq \overline{\omega}_{2,5}\|W\|_{X^0}\|V\|_{X^0}\,. \tag{7.106}$$

Moreover for $m \geq 2$ and $S \geq 5$ if ∂^α a space-time derivative such that $|\alpha| \leq m$, we have

$$\big|\big(\big[\partial^\alpha, L^\delta\big]W, V\big)\big| \leq \overline{\omega}_{m,S}\|W\|_{X^{k-1}}\|V\|_{X^0}\,. \tag{7.107}$$

Proof. To prove (7.105), as a preliminary, we shall first check the positivity of the second order operator $\mathcal{P} = -\nabla\cdot\big(A\nabla\big) + \alpha$. This is an elliptic operator as given by:

Lemma 7.29. *There exists $c > 0$ such that for every $u \in H^1(\mathbb{R}^2)$,*

$$\begin{aligned}\big(-\nabla\cdot(A(\nabla\eta)\nabla u) + \alpha u, u\big) &\geq (1+\|\nabla\eta\|_{L^\infty(\mathbb{R}^2)})^{-3}\|\nabla u\|^2 + \alpha\|u\|^2\\ &\geq c(1+\|\nabla\eta\|_{L^\infty(\mathbb{R}^2)})^{-3}\|u\|_{H^1(\mathbb{R}^2)}^2\,.\end{aligned}$$

Proof of Lemma 7.29. Since

$$A(\nabla\eta) = (1+|\nabla\eta|^2)^{-\frac{3}{2}}\begin{pmatrix} 1+(\partial_y\eta)^2 & -\partial_x\eta\partial_y\eta \\ -\partial_x\eta\partial_y\eta & 1+(\partial_x\eta)^2\end{pmatrix},$$

the statement follows from an integration by parts and the inequality

$$(\partial_y\eta)^2(\partial_x u)^2 + (\partial_x\eta)^2(\partial_y u)^2 - 2(\partial_x u)(\partial_y u)(\partial_x\eta)(\partial_y\eta) \geq 0.$$

This completes the proof of Lemma 7.29. □

Note that by Sobolev embedding, we have that

$$\|\nabla\eta^\delta\|_{L^\infty} \lesssim \|\eta\|_{H^4} + \|\nabla\eta^a\|_{L^\infty} \leq \bar{\omega}_{0,1}. \tag{7.108}$$

Consequently, with the notation $W = (W_1, W_2)$, we can use Lemma 7.16 and Lemma 7.29, to get

$$(L^\delta W, W) \geq \frac{1}{\overline{\omega}_{2,5}}\Big(\|W_1\|_{H^1}^2 + \Big\|\frac{|\nabla|}{(1+|\nabla|)^{\frac{1}{2}}}W_2\Big\|_{L^2}^2\Big) - \overline{\omega}_{2,5}\|W_1\|_{L^2}^2 + 2\Big(\partial_x W_1 - \nabla\cdot(W_1 v^\delta), W_2\Big).$$

Therefore (using in particular inequality (4.26)) we get

$$(L^\delta W, W) \geq \frac{1}{\overline{\omega}_{2,5}}\Big(\|W_1\|_{H^1}^2 + \||\nabla|^{\frac{1}{2}}W_2\|_{L^2}^2\Big) - \overline{\omega}_{2,5}\|W\|_{L^2}^2\,.$$

Using that $\|W_2\|_{H^{1/2}} \approx (\|W_2\|_{L^2} + \||\nabla|^{\frac{1}{2}} W_2\|_{L^2})$, we obtain that

$$(L^\delta W, W) \geq \frac{1}{\overline{\omega}_{2,5}} \Big(\|W_1\|_{H^1}^2 + \|W_2\|_{H^{\frac{1}{2}}}^2 \Big) - \overline{\omega}_{2,5} \|W\|_{L^2}^2$$

which in turn implies the claimed inequality.

To prove (7.106), let us write $W = (W_1, W_2) \in H^1 \times H^{\frac{1}{2}}$ and $V = (V_1, V_2) \in H^1 \times H^{\frac{1}{2}}$, then we have

$$\begin{aligned} (L^\delta W, V) &= (\mathcal{P}^\delta W_1, V_1) + (G^\delta W_2, V_2) + \big(((v^\delta \cdot \nabla Z^\delta) + (\partial_t - \partial_x) Z^\delta) \, W_1, V_1 \big) \\ &\quad - (\nabla \cdot (v^\delta W_1) - \partial_x W_1, V_2) - (\nabla \cdot (v^\delta V_1) - \partial_x V_1, W_2) \, . \end{aligned}$$

We now estimate each term in the above decomposition. Coming back to the definition of $\mathcal{P}$, using an integration by parts and (7.108) we have

$$|(\mathcal{P}^\delta W_1, V_1)| \leq \overline{\omega}_{2,5} \|W_1\|_{H^1} \|V_1\|_{H^1} \, .$$

Next using Lemma 7.16, we also have that

$$|(G^\delta W_2, V_2)| \leq \overline{\omega}_{2,5} \|W_2\|_{H^{\frac{1}{2}}} \|V_2\|_{H^{\frac{1}{2}}} .$$

Next, we can use the Sobolev embedding and Lemma 7.22 to get

$$\Big| \big(((v^\delta \cdot \nabla Z^\delta) + (\partial_t - \partial_x) Z^\delta) \, W_1, V_1 \big) \Big| \leq \overline{\omega}_{2,5} \|W_1\|_{H^1} \|V_1\|_{H^1} .$$

The last two terms in the decomposition can be estimated by invoking the inequality

$$|(\nabla \cdot (v^\delta V_1) - \partial_x V_1, W_2)| \leq \overline{\omega}_{2,5} \|V_1\|_{H^1} \|W_2\|_{H^{\frac{1}{2}}}$$

which is a direct consequence of the Sobolev embedding and Lemma 7.22. This completes the proof of the first inequality in our statement. Let us now turn to the proof of the second inequality. We can first write

$$\begin{aligned} ([\partial^\alpha, L^\delta] W, V) &\leq |([\partial^\alpha, \mathcal{P}^\delta] W_1, V_1)| + |([\partial^\alpha, G^\delta] W_2, V_2)| \\ &\quad + |\big([\partial^\alpha, ((v^\delta \cdot \nabla Z^\delta) + (\partial_t - \partial_x) Z^\delta)] W_1, V_1 \big)| \\ &\quad + |(\nabla \cdot ([\partial^\alpha, v^\delta] W_1), V_2)| + |(\nabla \cdot ([\partial^\alpha, v^\delta] V_1), W_2)| \, . \end{aligned}$$

Now each term in the above decomposition can be estimated as in the proof of the first inequality by taking the advantage of the commutator structure and thus by using Lemma 7.23. This completes the proof of Proposition 7.28. □

7.8. **Energy estimates for the main part of** (7.67)**.** Recall also that $U(0) = 0$. The goal of this section is to prove the following statement.

Proposition 7.30. *For $m \geq 2$, $S \geq 5$, a smooth solution of (7.69) satisfy the estimate*

$$\begin{aligned} \|U(t)\|_{X^{m+3}}^2 \leq \omega \Big(\|R^{ap}\|_{X_t^{m+3}} + \|V^a\|_{\mathcal{W}_t^{m+S}} + \|U\|_{X_t^{m+3}} \Big) \\ \times \Big(\|R^{ap}\|_{X_t^{m+3}}^2 + \int_0^t \big(\|U(\tau)\|_{X^{m+3}}^2 + \|F(\tau)\|_{X^{m+3}}^2 \big) d\tau \Big). \end{aligned}$$

Proof of Proposition 7.30. It is more convenient to perform the energy estimate on the equation (7.71) satisfied by $W_{ijk} = PU_{ijk}$. Since thanks to Lemma 7.22, for $S \geq 5$, we have

$$\|W_{ijk}\|_{X^m} \leq \overline{\omega}_{m,S} \|U\|_{X^{m+3}} \tag{7.109}$$

and

$$\|U\|_{X^{m+3}} \leq \overline{\omega}_{m,S} \Big(\sum_{i,j,k} \|W_{ijk}\|_{X^m} + \|U\|_{X^3} \Big), \tag{7.110}$$

it is equivalent to estimate W_{ijk} or U_{ijk}. We thus consider, for $|\alpha| \le m$, the equation solved by $\partial^\alpha W_{ijk}$. Namely

$$\partial_t \partial^\alpha W_{ijk} = J\Big(L^\delta \partial^\alpha W_{ijk} + [\partial^\alpha, L^\delta] W_{ijk} - \partial^\alpha JPJ\big((\mathcal{Q}^{ijk})^\delta - (\mathcal{Q}^{ijk})^a\big)\Big) + \partial^\alpha (PF_{ijk}). \tag{7.111}$$

We take the L^2 scalar product of (7.111) with

$$\mathcal{M} \equiv L^\delta \partial^\alpha W_{ijk} + [\partial^\alpha, L^\delta] W_{ijk} - \partial^\alpha JPJ\big((\mathcal{Q}^{ijk})^\delta - (\mathcal{Q}^{ijk})^a\big).$$

From the skew symmetry of J and the symmetry of L^δ, we get the identity

$$\frac{d}{dt}\Big(\frac{1}{2}\big(\partial^\alpha W_{ijk}, L^\delta \partial^\alpha W_{ijk}\big) + \mathcal{I}^\alpha_{ijk}\Big) = \mathcal{J}^\alpha_{ijk} \tag{7.112}$$

where

$$\mathcal{I}^\alpha_{ijk} = \big(\partial^\alpha W_{ijk}, [\partial^\alpha, L^\delta] W_{ijk}\big) - \big(\partial^\alpha W_{ijk}, \partial^\alpha\big(JPJ\big((\mathcal{Q}^{ijk})^\delta - (\mathcal{Q}^{ijk})^a\big)\big)\big), \tag{7.113}$$

$$\begin{aligned}\mathcal{J}^\alpha_{ijk} = {}& \frac{1}{2}\big(\partial^\alpha W_{ijk}, [\partial_t, L^\delta]\partial^\alpha W_{ijk}\big) + \big(\partial^\alpha W_{ijk}, \partial_t [\partial^\alpha, L^\delta] W_{ijk}\big) \\ & -\big(\partial^\alpha W_{ijk}, \partial_t \partial^\alpha\big(JPJ\big((\mathcal{Q}^{ijk})^\delta - (\mathcal{Q}^{ijk})^a\big)\big) + \big(\mathcal{M}, \partial^\alpha(PF_{ijk})\big).\end{aligned} \tag{7.114}$$

For the proof of Proposition 7.30, we shall estimate separately all the terms arising in the energy identity (7.112) in the following sequence of Lemmas.

We first have the following estimate for $\mathcal{I}^\alpha_{ijk}$ and $\mathcal{J}^\alpha_{ijk}$:

Lemma 7.31. *For $m \ge 2$, $S \ge 5$, we have the estimates*

$$|\mathcal{J}^\alpha_{ijk}| \le \overline{\omega}_{m,S}\big(\|U\|_{X^{m+3}}\|F_{ijk}\|_{X^m} + \|U\|^2_{X^{m+3}}\big), \quad |\alpha| \le m \tag{7.115}$$

and

$$|\mathcal{I}^\alpha_{ijk}| \le \overline{\omega}_{m,S}\|U\|_{X^{m+3}}\|U\|_{X^{m+2}}, \quad |\alpha| \le m. \tag{7.116}$$

Proof. Let us start with the estimate of $\mathcal{I}^\alpha_{ijk}$. In view of (7.113), we can write

$$\mathcal{I}^\alpha_{ijk} = I_1 - I_2.$$

From (7.107), in Proposition 7.28 and Cauchy-Schwarz, we immediately get

$$|I_1| \le \overline{\omega}_{m,S}\|U\|_{X^{m+3}}\|U\|_{X^{m+2}}.$$

To estimate the second term, we can expand

$$\begin{aligned} I_2 &= \big(\partial^\alpha W_{ijk}[1], \partial^\alpha\big((\mathcal{Q}^{ijk}_2)^\delta - (\mathcal{Q}^{ijk}_2)^a\big)\big) - \big(\partial^\alpha W_{ijk}[1], \partial^\alpha\big(Z^\delta\big((\mathcal{Q}^{ijk}_1)^\delta - (\mathcal{Q}^{ijk}_1)^a\big)\big)\big) \\ &\quad + \Big(\partial^\alpha W_{ijk}[2], \partial^\alpha\big((\mathcal{Q}^{ijk}_1)^\delta - (\mathcal{Q}^{ijk}_1)^a\big)\big)\Big) \\ &= I_2^1 + I_2^2 + I_2^3 \end{aligned}$$

by using the notation $W_{ijk} = (W_{ijk}[1], W_{ijk}[2])^t$. A direct application of Proposition 7.27 yields

$$|I_2^1| \le \|\partial^\alpha W^{ijk}[1]\|_{H^1}\|(\mathcal{Q}^{ijk}_2)^\delta - (\mathcal{Q}^{ijk}_2)^a\|_{H^{-1}} \le \overline{\omega}_{m,S}\|U\|_{X^{m+3}}\|U\|_{X^{m+2}}$$

and

$$|I_2^3| \le \|\partial^\alpha W^{ijk}[2]\|_{H^{\frac{1}{2}}}\|(\mathcal{Q}^{ijk}_1)^\delta - (\mathcal{Q}^{ijk}_1)^a\|_{H^{-\frac{1}{2}}} \le \overline{\omega}_{m,S}\|U\|_{X^{m+3}}\|U\|_{X^{m+2}}.$$

Note that we have used (7.109) for the second part of the estimates. For the estimate of I_2^2, we can use the following lemma:

Lemma 7.32. *For $l \geq 2$, $l \leq m+1$, $m \geq 2$ and $S \geq 5$, we have the estimates*

$$\|\langle\partial\rangle^{m+2}\big(Z^\delta - Z^a\big)\|_{H^\sigma} \leq \overline{\omega}_{m,S}\|\langle\partial\rangle^{m+2}U\|_{H^{\sigma+1}}, \quad \sigma = -\frac{1}{2},\, 0,\, \frac{1}{2} \tag{7.117}$$

and

$$\big|\big(\partial^\beta\big(Z^\delta\big((\mathcal{Q}_1^{ijk})^\delta - (\mathcal{Q}_1^{ijk})^a\big)\big), F\big)\big| \leq \overline{\omega}_{m,S}\|U\|_{X^{l+2}}\|F\|_{H^1}, \quad |\beta| \leq l. \tag{7.118}$$

Let us postpone the proof of this last lemma. By a direct application of it with $F = \partial^\alpha W^{ijk}[1]$, $\beta = \alpha$, $l = m$ and a new use of (7.109), we also get

$$|I_2^2| \leq \overline{\omega}_{m,S}\|U\|_{X^{m+3}}\|U\|_{X^{m+2}}.$$

This proves (7.116).

Let us turn to the estimate of $\mathcal{J}_{ijk}^\alpha$. In view of (7.114), we set

$$\mathcal{J}_{ijk}^\alpha = J_1 + J_2 - J_3 + J_4.$$

By using (7.107) in Proposition 7.28 and (7.109), we infer

$$|J_1| \leq \ \overline{\omega}_{m,S}\|W_{ijk}\|_{X^m}^2 \leq \overline{\omega}_{m,S}\|U\|_{X^{m+3}}^2\,. \tag{7.119}$$

In order to estimate J_2, it suffices to write $\partial_t[\partial^\alpha, L^\delta] = [\partial_t\partial^\alpha, L^\delta] + [L^\delta, \partial_t]\partial^\alpha$ and to apply (7.107). This yields

$$|J_2| \leq \overline{\omega}_{m,S}\|U\|_{X^{m+3}}^2.$$

To estimate J_3, we first expand

$$\begin{aligned} J_3 &= \big(\partial^\alpha W_{ijk}[1], \partial_t\partial^\alpha\big((\mathcal{Q}_2^{ijk})^\delta - (\mathcal{Q}_2^{ijk})^a\big)\big) - \big(\partial^\alpha W_{ijk}[1], \partial_t\partial^\alpha\big(Z^\delta\big(\mathcal{Q}_1^{ijk})^\delta - (\mathcal{Q}_1^{ijk})^a\big)\big)\big) \\ &\quad + \big(\partial^\alpha W_{ijk}[2], \partial_t\partial^\alpha\big((\mathcal{Q}_1^{ijk})^\delta - (\mathcal{Q}_1^{ijk})^a\big)\big) \\ &= J_3^1 + J_3^2 + J_3^3. \end{aligned}$$

The estimate of J_3^1 and J_3^2 follows by a direct application of Proposition 7.27 (changing m into $m+1$) while the estimate of J_3^2 is a consequence of the estimate (7.118) of Lemma 7.32.

It remains to estimate J_4. Let us write

$$\begin{aligned} \big(\mathcal{M}, \partial^\alpha(PF_{ijk})\big) &= \big(L^\delta\partial^\alpha W_{ijk}, \partial^\alpha(PF_{ijk})\big) + \big([\partial^\alpha, L^\delta]W_{ijk}, \partial^\alpha(PF_{ijk})\big) \\ &\quad - \big(\partial^\alpha\big(JPJ\big((\mathcal{Q}^{ijk})^\delta - (\mathcal{Q}^{ijk})^a\big)\big), \partial^\alpha(PF_{ijk})\big). \end{aligned}$$

We can estimate the two first terms by using (7.106), (7.107) in Proposition 7.28, while the estimate of the last term follows by using again Proposition 7.27 and Lemma 7.32. This ends the proof of Lemma 7.31, it just remains to prove Lemma 7.32. ☐

Proof of Lemma 7.32. We start with the proof of (7.117). By using the same decomposition of Z as in the proof of Lemma 7.22 and in particular (7.79), we see that the proof follows from the following estimate on G:

$$\|\langle\partial\rangle^{m+2}\big(G^\delta\varphi - G^a\varphi\big)\|_{H^\sigma} \leq \overline{\omega}_{m,S}\|\langle\partial\rangle^{m+2}\varphi\|_{H^{\sigma+1}}, \quad \sigma = -\frac{1}{2},\, 0,\, \frac{1}{2}.$$

Since

$$G^\delta\varphi - G^a\varphi = \int_0^1 D_\eta G[\eta^a + s\eta]\varphi\cdot\eta\,ds$$

the needed estimate for $\sigma = -1/2$ and $\sigma = 1/2$ is a consequence of the refined estimate (7.11) in Remark 7.4. The estimate for $\sigma = 0$ follows by interpolation.

To prove (7.118), we first write for $\beta_1 + \beta_2 = \beta$

$$|(\partial^{\beta_1}\big(Z^\delta)\partial^{\beta_2}\big((\mathcal{Q}_1^{ijk})^\delta - (\mathcal{Q}_1^{ijk})^a\big), F\big)| \leq \|\partial^{\beta_2}\big((\mathcal{Q}_1^{ijk})^\delta - (\mathcal{Q}_1^{ijk})^a\big)\|_{H^{-\frac{1}{2}}}\|\partial^{\beta_1}(Z^\delta)F\|_{H^{\frac{1}{2}}}\,.$$

The first term in the right hand-side of the above inequality can be estimated by using the estimate (7.101) of Proposition 7.27 while for the second one, we use Lemma 7.12 to get

$$\begin{aligned}\|\partial^{\beta_1}(Z^\delta)F\|_{H^1} &\leq \|\partial^{\beta_1}(Z^\delta - Z^a)F\|_{H^1} + \|\partial^{\beta_1}(Z^a)F\|_{H^1} \\ &\leq \|\partial^{\beta_1}(Z^\delta - Z^a)F\|_{H^1} + \omega(\|V^a\|_{\mathcal{W}^{m+S}})\|F\|_{H^1}.\end{aligned}$$

Next, we invoke the inequality

$$\|uv\|_{H^1(\mathbb{R}^2)} \leq C\|u\|_{H^{\frac{3}{2}}(\mathbb{R}^2)}\|v\|_{H^1(\mathbb{R}^2)}$$

to get

$$\|\partial^{\beta_1}(Z^\delta - Z^a)F\|_{H^1} \leq C\,\|\partial^{\beta_1}(Z^\delta - Z^a)\|_{H^{\frac{3}{2}}}\|F\|_{H^1} \leq C\,\|\langle\partial\rangle^{l+1}(Z^\delta - Z^a)\|_{H^{\frac{1}{2}}}\|F\|_{H^1}$$

and hence

$$\|\partial^{\beta_1}(Z^\delta - Z^a)F\|_{H^1} \leq \overline{\omega}_{m,S}\|U\|_{X^{m+3}}\|F\|_{H^1} \leq \overline{\omega}_{m,S}\|F\|_{H^1}$$

thanks to (7.117). This completes the proof of Lemma 7.32. $\square$

We shall consider in addition to the energy identity (7.112) the following estimate:

Lemma 7.33. *Let U be a solution of (7.2) then for $m \geq 2$ and $S \geq 5$,*

$$\frac{d}{dt}\|U(t)\|_{X^3}^2 \leq \omega\Big(\|R^{ap}\|_{X^3} + \|V^a\|_{\mathcal{W}_t^{m+S}} + \|U\|_{X_t^{m+3}}\Big)\|U\|_{X^{m+3}}^2.$$

Moreover, we also have the estimate

$$\|\partial_t^l U\|_{L^2}^2 \leq \overline{\omega}_{m,S}\Big(\|\,|\nabla|\partial_t^{l-1}U\|_{X^0}^2 + \|U\|_{X^{m+2}}^2 + \|R^{ap}\|_{X^m}^2\Big), \quad 4 \leq l \leq m+3. \tag{7.120}$$

This Lemma will be crucial in order to get the claimed estimate of Proposition 7.30 from the identity (7.112). The first estimate will be used to compensate the fact that L^δ does not control the low frequencies or equivalently the L^2 norm (see (7.105)). The second estimate will be used to prove that by a suitable summation of the identities (7.112) we get a control of the X^{m+3} norm of W. Note that this second estimate basically states that thanks to the equation, we can replace a time derivative by space derivatives.

Proof. For the estimate of the L^2 norm of U, it suffices to multiply (7.2) by U, integrate over $\mathbb{R}^2$ and use very crude estimates of all the terms. We then proceed in a similar way by taking at most three derivatives of the equation to get the first claimed estimate.

To get the second estimate, we apply ∂_t^{l-1} to (7.67). We get by taking the L^2 norm of the obtained equation that

$$\|\partial_t^l U\|_{L^2} \leq \overline{\omega}_{m,S}\Big(\|\partial_t^{l-1}\eta\|_{H^2} + \|\partial_t^{l-1}\varphi\|_{H^1} + \|U\|_{X^{m+2}}\Big) + \|R^{ap}\|_{X^m}.$$

We shall not give the details of this estimate which can be obtained by using the same kind of estimates as in Propositions 7.3 and 7.13 as previously. Note that in particular the term involving φ arises in this form by using (7.117) for $\sigma = 0$. The idea behind this estimate is simple, the equation allows to replace one time derivative by one space derivative for φ and two space derivatives for η. Next, since we have that

$$\|\,|\nabla|\big(|\nabla|\partial_t^{l-1}\eta\big)\|_{L^2} \leq C\||\nabla|\partial_t^{l-1}U\|_{X^0}$$

and by standard interpolation that

$$\|\nabla\partial_t^{l-1}C\varphi\|_{L^2}^2 \leq \|\,|\nabla|^{\frac{1}{2}}\big(|\nabla|\partial_t^{l-1}\big)\varphi)\|_{L^2}\,\|\,|\nabla|^{\frac{1}{2}}\partial_t^{l-1}\varphi\|_{L^2} \leq C\|U\|_{X^{m+3}}\|U\|_{X^{m+2}}, \tag{7.121}$$

the result follows. This completes the proof of Lemma 7.33. $\square$

To prove Proposition 7.30, in view of (7.112), we shall use the energy

$$E_\alpha(t) \equiv \sum_{i,j,k} \Big(\frac{1}{2} \big(\partial^\alpha W_{ijk}, L^\delta \partial^\alpha W_{ijk} \big) + \mathcal{I}^\alpha_{ijk} \Big).$$

We also define for $m \geq 2$, $1 \leq l \leq m$, $\tau \in [0, t]$

$$E_{l,m}(\tau) = \sum_{1 \leq |\alpha| \leq l \ \alpha' \neq 0} E_\alpha(\tau) + \Gamma \sum_{1 \leq |\alpha| \leq l, \, \alpha' = 0} E_\alpha(\tau)$$

where $\alpha = (\alpha_0, \alpha')$ with $\alpha' = (\alpha_1, \alpha_2)$ and $\Gamma > 0$ (which depends on m and t) will be carefully chosen. The aim of the following sequence of lemmas is to obtain the positivity of the energy in order to deduce an estimate from (7.112). We first have the following :

Lemma 7.34. *There exists $\Gamma > 0$ such that for every $\tau \in [0, t]$, we have*

$$E_{l,m}(\tau) \geq \frac{1}{\overline{\omega}_{m,S,t}} \|U(\tau)\|^2_{X^{l+3}} - \overline{\omega}_{m,S,t} \|U(\tau)\|^2_{X^{l+2}}$$

where $\overline{\omega}_{m,S,t} = \omega \big(\|V^a\|_{\mathcal{W}^{m+S}_t} + \|U\|_{X^{m+3}_t} \big)$.

Proof. We first consider the case that $\alpha' \neq 0$, i.e. $\partial^\alpha \neq \partial^l_t$. In this case, we have from (7.105) of Proposition 7.28 and Lemma 7.31 that

$$E_\alpha(\tau) \geq \frac{1}{\overline{\omega}_{m,S}} \|\partial^\alpha W(\tau)\|^2_{X^0} - \|\partial^\alpha W(\tau)\|^2_{L^2} - \overline{\omega}_{m,S} \|U(\tau)\|_{X^{l+3}} \|U(\tau)\|_{X^{l+2}}.$$

Now, since ∂^α contains at least one space derivative, we can write for $|\beta| = |\alpha| - 1$ that

$$\|\partial^\alpha W(\tau)\|^2_{L^2} \lesssim \|\nabla \partial^\beta W(\tau)\|^2_{L^2} \lesssim \|U(\tau)\|_{X^{l+3}} \|U(\tau)\|_{X^{l+2}}$$

by using again the interpolation inequality (7.121). Consequently, by using the Young inequality and (7.110), we find that

$$\sum_{\alpha' \neq 0} E_\alpha(\tau) \geq \frac{1}{\overline{\omega}_{m,S,t}} \sum_{\alpha' \neq 0} \|\partial^\alpha U(\tau)\|^2_{X^0} - \overline{\omega}_{m,S,t} \|U(\tau)\|_{X^{l+2}} \|U(\tau)\|_{X^{l+3}}, \quad \forall \tau \in [0, t]. \tag{7.122}$$

Now, let us consider the case that $\partial^\alpha = \partial^l_t$ i.e. $\alpha' = 0$. By the same consideration as above, we first get

$$E_\alpha(\tau) \geq \frac{1}{\overline{\omega}_{m,S,t}} \|\partial^l_t W(\tau)\|^2_{X^0} - \|\partial^l_t W(\tau)\|^2_{L^2} - \overline{\omega}_{m,S,t} \|U(\tau)\|_{X^{l+3}} \|U(\tau)\|_{X^{l+2}}.$$

Next, from (7.109) and (7.120), we get

$$\|\partial^l_t W(\tau)\|^2_{L^2} \leq \overline{\omega}_{m,S,t} \big(\sum_{\alpha' \neq 0} \|\partial^\alpha U(\tau)\|^2_{X^0} + \|U(\tau)\|^2_{X^{m+2}} \big).$$

This yields

$$E_\alpha(\tau) \geq \frac{1}{\overline{\omega}_{m,S,t}} \|\partial^m_t W\|^2_{X^0} - \overline{\omega}_{m,S,t} \big(\sum_{\alpha' \neq 0} \|\partial^\alpha U(\tau)\|^2_{X^0} + \|U\|^2_{X^{l+2}} \big), \quad \forall \tau \in [0, t]. \tag{7.123}$$

Consequently, we can add (7.122) times Γ sufficiently large to (7.123) and use the Young inequality to get the result. This completes the proof of Lemma 7.34. $\square$

Let us finally set for $\tau \in [0, t]$

$$E_m(\tau) = \sum_{1 \leq l \leq m} \Gamma^{m-l} E_{l,m}(\tau) + \Gamma \|U(\tau)\|^2_{X^3}$$

for Γ possibly larger to be chosen. We have

Lemma 7.35. *For every $t > 0$, there exists Γ such that for every $\tau \in [0, t]$, we have*

$$E_m(\tau) \geq \frac{1}{\overline{\omega}_{m,S,t}} \|U(\tau)\|^2_{X^{m+3}}.$$

Proof. We get from Lemma 7.34 that

$$\sum_{1 \leq l \leq m} \Gamma^{m-l} E_{l,m}(\tau) \geq \frac{1}{\overline{\omega}_{m,S,t}} \|U(\tau)\|^2_{X^{m+3}} + \sum_{1 \leq l \leq m-1} \Big(\frac{\Gamma^{m-l}}{\overline{\omega}_{m,S,t}} - \Gamma^{m-l-1} \overline{\omega}_{m,S,t} \Big) \|U(\tau)\|_{X^{l+3}} + \big(\Gamma - \overline{\omega}_{m,S,t}\big) \|U(\tau)\|^2_{X^3}.$$

Consequently, for Γ so that $\Gamma \geq \overline{\omega}_{m,S,t}$ we get

$$E_m(\tau) \geq \frac{1}{\overline{\omega}_{m,S,t}} \|U(\tau)\|^2_{X^{m+3}}.$$

This completes the proof of Lemma 7.35. □

We are now in position to end the proof of Proposition 7.30. By using the identity (7.112) and Lemma 7.31, we get

$$\Big| \frac{d}{dt} E_m(\tau) \Big| \leq \omega \Big(\|R^{ap}\|_{X_t^{m+S}} + \|V^a\|_{\mathcal{W}_t^{m+S}} + \|U\|_{X_t^{m+3}} \Big) \Big(\|U(\tau)\|^2_{X^{m+3}} + \|F_{ijk}(\tau)\|^2_{X^m} \Big), \quad 0 \leq \tau \leq t\,.$$

Moreover, from the equation solved by U, we obtain that at the initial time

$$|E_m(0)| \leq \omega \Big(\|R^{ap}\|_{X_0^{m+3}} + \|V^a\|_{\mathcal{W}_0^{m+S}} + \|U\|_{X_0^{m+3}} \Big) \|R^{ap}\|^2_{X_0^{m+3}}\,.$$

Consequently, we can integrate in time for $\tau \in [0, t]$ and use Lemma 7.35 to end the proof of Proposition 7.30. □

7.9. **Proof of the energy estimate: proof of Theorem 7.1.** It suffices to combine Proposition 7.30 and Proposition 7.17.

7.10. **Final argument. End of the proof of Theorem 1.4.** In this section we complete the proof of Theorem 1.4. From Theorem 7.1, we have for the solution of (7.67) with initial data $V^a(0) = Q + \delta U^a(0)$ that

$$\|U(t)\|^2_{X^{m+3}} \leq \omega \Big(\|R^{ap}\|_{X_t^{m+3}} + \|V^a\|_{\mathcal{W}_t^{m+S}} + \|U\|_{X_t^{m+3}} \Big) \times \Big(\|R^{ap}\|^2_{X_t^{m+3}} + \int_0^t \big(\|U(\tau)\|^2_{X^{m+3}} + \|R^{ap}(\tau)\|^2_{X^{m+3}} \big) d\tau \Big). \tag{7.124}$$

Using (7.124) and some standard arguments (see the next section), we can define local strong solutions of the water waves equation with data $Q + \delta U^a$. Note that for the argument providing this small time existence, the specific structure of R^{ap} is not of importance, one only needs to know that it belongs to Sobolev spaces. We now show that the estimates on R^{ap} provided by Proposition 6.3 allow to extend the solution on much longer times (sufficiently long so that we see the instability). Thanks to Proposition 6.3 (with s changed into m and thus m changed into p), we have the bounds

$$\|R^{ap}(t)\|_{X^{m+3}} \leq C_{M,m} \delta^{M+3} \frac{e^{(M+3)\sigma_0 t}}{(1+t)^{\frac{M+3}{2p}}},$$

provided $0 \leq t \leq T^\delta$, $0 \leq \delta < \delta_0 \ll 1$ with T^δ such that

$$\frac{e^{\sigma_0 T^\delta}}{(1+T^\delta)^{\frac{1}{2p}}} = \frac{\kappa}{\delta},$$

where $\kappa \in (0,1)$ is a small number to be chosen later, independantly of $\delta \in (0,\delta_0)$. Coming back to (7.124), we infer that

$$\|U(t)\|^2_{X^{m+3}} \leq \omega\big(C + \|U\|_{X^{m+3}_t} + \kappa C_{M,m}\big)\Big(\int_0^t \|U(\tau)\|^2_{X^{m+3}} d\tau + \frac{\delta^{2(M+3)} e^{2(M+3)\sigma_0 t}}{(1+t)^{\frac{M+3}{p}}}\Big), \tag{7.125}$$

as far as $0 \leq t \leq T^\delta$. Let us define T^* as

$$T^* = \sup\{T \,:\, T \leq T^\delta, \text{ and } \forall\, t \in [0,T], \|U\|_{X^{m+3}_T} \leq 1,\, 1 - \|\eta\|_{L^\infty} - \|\eta^a\|_{L^\infty} > 0\}.$$

Observe that T is well-defined, at least for $\delta \ll 1$. Using (7.125), we obtain that for $0 \leq t < T^\star$,

$$\|U(t)\|^2_{X^{m+3}} \leq \omega(C + \kappa C_{M,m})\Big(\int_0^t \|U(\tau)\|^2_{X^{m+3}} d\tau + \frac{\delta^{2(M+3)} e^{2(M+3)\sigma_0 t}}{(1+t)^{\frac{M+3}{p}}}\Big). \tag{7.126}$$

We take an integer M large enough so that $2(M+3)\sigma_0 - \omega(C) \geq 20$. At this place we fix the value of M. We then choose κ small enough so that $1 > \omega(C + \kappa C_{M,m}) - \omega(C)$. Such a choice of κ is possible thanks to the continuity assumption on ω. We also observe that for $A \geq 1$ and $\rho \geq 0$, there exists C such that for every $t \geq 0$, we have the inequality

$$\int_0^t \frac{e^{A\tau}}{(1+\tau)^\rho} d\tau \leq C \frac{e^{At}}{(1+t)^\rho}\,. \tag{7.127}$$

Thanks to (7.126), (7.127) and the choice of M and κ, we can apply a bootstrap argument and the Gronwall lemma, we infer that $U(t)$ is defined for $t \in [0, T^\delta]$ and that

$$\sup_{0 \leq t \leq T^\delta} \|U(t)\|_{X^{m+3}} \leq C_{M,m} \kappa^{M+3}\,. \tag{7.128}$$

The bound (7.128) implies in particular that

$$\|U(T^\delta)\|_{L^2(\mathbb{R}^2)} \leq C_{M,m}\kappa^{M+3}\,.$$

Let I be the time interval involved in the definition of U^0 (see (6.5)). Let us fix $\theta \in C_0^\infty(\mathbb{R})$ which equals one on I and which vanishes near zero. Let Π be a Fourier multiplier on $\mathbb{R}^2_{\xi_1,\xi_2}$ with symbol $\theta(\xi_2)$ (i.e. cutting the zero frequency in y). The map Π is bounded on $L^2(\mathbb{R}^2)$. We also have that $\Pi(U^0) = U^0$. Therefore, using Proposition 6.1 and Proposition 6.3, we obtain that for every $t \geq 0$

$$\begin{aligned}
\|\Pi(\delta U^a(t))\|_{L^2(\mathbb{R}^2)} &\geq c\delta \frac{e^{\sigma_0 t}}{(1+t)^{\frac{1}{2p}}} - \sum_{j=1}^{M+1} \delta^{j+1} \|\Pi(U^i(t))\|_{L^2(\mathbb{R}^2)} \\
&\geq c\delta \frac{e^{\sigma_0 t}}{(1+t)^{\frac{1}{2p}}} - C_M \sum_{j=1}^{M+1} \delta^{j+1} \frac{e^{(j+1)\sigma_0 t}}{(1+t)^{\frac{j+1}{2p}}}\,.
\end{aligned}$$

Thus for $\kappa \ll 1$,

$$\|\Pi\big(\delta U^a(T^\delta)\big)\|_{L^2(\mathbb{R}^2)} \geq \frac{c\kappa}{2}, \quad \forall\, t \geq 0.$$

Observe that for every $a \in \mathbb{R}$, $\Pi(Q(\cdot - a)) = 0$. Recall that the true solution U^δ with data $Q + \delta U^0(0)$ is decomposed as $U^\delta = Q + \delta U^a + U$. In particular at time zero U^δ is δ close to Q in any $H^s(\mathbb{R}^2)$. On the other hand, for every $a \in \mathbb{R}$, we can write

$$\begin{aligned}
\|U^\delta(T^\delta,\cdot) - Q(\cdot - a)\|_{L^2} &\geq c\|\Pi(U^\delta(T^\delta,\cdot) - Q(\cdot - a))\|_{L^2} \\
&= c\|\Pi(U^\delta(T^\delta,\cdot) - Q(\cdot))\|_{L^2} \\
&= c\|\Pi(\delta U^a(T^\delta) + U(T^\delta))\|_{L^2} \\
&\geq c\kappa - C_M \kappa^{M+3} \geq \frac{c\kappa}{2},
\end{aligned}$$

provided $\kappa \ll 1$ (independantly of δ). This completes the proof of Theorem 1.4.

8. Sketch of the existence proof

In this section, we sketch an existence proof by a vanishing viscosity type method for (7.2). The local existence of a smooth solution for (6.1) in Sobolev spaces (which corresponds to the study of (7.2) when $V^a = 0$ was already obtained in [27] by using the Nash-Moser iteration scheme or in [4] by using a subtle Lagrangian type formulation of the problem, we also refer to the work [11], [34] for the case with vorticity.

The aim of the following is to sketch a simple proof which avoids the use of a complicated iteration scheme.

For a positive number ν, we consider the system

$$\partial_t U = \mathcal{F}(U^\delta) - \mathcal{F}(V^a) - \nu\, \Delta^2 U - R^{ap}. \tag{8.1}$$

By standard arguments, for every positive ν, one can get a solution U^ν of (8.1) in $C([0, T^\nu], H^s)$ for some $T^\nu > 0$ when $s \geq s_0$ is sufficiently large ($s > 3$). Indeed, we can use Propositions 7.3, 7.13 and Remark 7.11 in the classical Sobolev framework (i.e. without the time derivative, since the time is only parameter in these propositions, their statement remain obviously valid if one replace ∂ by ∇). From these estimates, the local existence for (8.1) follows from Duhamel formula and the Banach fixed point Theorem. We also get that the solution can be continued as long as the H^{s_0} norm for some s_0 sufficiently large remains bounded.

The next step is to prove that the existence time of U^ν is uniform in ν i.e. we have to prove that for some $T > 0$, we have $T^\nu \geq T > 0$, for every $\nu \in (0, 1]$. Towards this, it suffices to prove that the H^s norm of U^ν cannot blow-up on $[0, T]$ for some positive T independent of ν. The main idea is that the estimate of Theorem 7.1 still holds for (8.1) uniformly in ν. By using the same transformation as in Section 7.4.2, we find that (7.71) is changed into

$$\partial_t W_{ijk} = J\big(L^\delta W_{ijk} - JPJ\big((\mathcal{Q}^{ijk})^\delta - (\mathcal{Q}^{ijk})^a\big)\big) + PF_{ijk}, -\nu\, \Delta^2 W^{ijk} + \mathcal{R}^\nu \tag{8.2}$$

with $W_{ijk} = PU_{ijk}$ (for the proof of the energy estimate, we shall denote U^ν by U for the sake of clarity) and the subprincipal term $\mathcal{Q}^{ijk}$ and F which contains the semilinear terms $\mathcal{G}^{ijk}$ are the same as previously (in particular, F is given by (7.70)). In particular they still satisfy with ω independent of ν the estimates of Proposition 7.27 and Proposition 7.17. The only new terms that show up are the diffusion term $-\nu\, \Delta^2$ and $\mathcal{R}^\nu$ which is defined as

$$\mathcal{R}^\nu = \begin{pmatrix} 0 \\ \nu[Z^\delta, \Delta^2] U_{ijk}[1] \end{pmatrix} = \begin{pmatrix} 0 \\ \nu[Z^\delta, \Delta^2] W_{ijk}[1] \end{pmatrix}. \tag{8.3}$$

Our goal is to perform energy estimates, uniform in $\nu \in (0, 1]$ to (8.2). Such estimates follow the same lines as the energy estimates we performed for $\nu = 0$. One should take care of the terms coming from the ν dependence. Our situation is slightly different from the classical one because of the presence of $\mathcal{R}^\nu$ coming form the transformation $W_{ijk} = PU_{ijk}$. The point is that, thanks to the parabolic term $-\nu\Delta^2$, the contributions of the terms containing ν can be put in norms containing two more derivatives with respect the energy level. For instance, the estimates on Z^δ, we already established are strong enough to control the contribution of $\mathcal{R}^\nu$. More precisely, thanks to Lemma 7.22, we can estimate Z^δ in norms containing two more derivatives with respect to the energy level. This essentially explains the approach to energy estimates for the equation (8.2).

By applying ∂^α to (8.2), we deduce the energy identity which is the straightforward generalization of (7.112)

$$\frac{d}{dt}\Big(\frac{1}{2}\big(\partial^\alpha W_{ijk}, L^\delta \partial^\alpha W_{ijk}\big) + \mathcal{I}^\alpha_{ijk}\Big) = \mathcal{J}^\alpha_{ijk} + \mathcal{D}^\nu$$

where $\mathcal{I}^\alpha_{ijk}$ and $\mathcal{J}^\alpha_{ijk}$ are still given by (7.113), (7.114) and

$$\begin{aligned}\mathcal{D}^\nu &= -\nu\left(\Delta^2\partial^\alpha W_{ijk}, L^\delta\partial^\alpha W_{ijk}\right) + \left(\partial^\alpha\mathcal{R}^\nu, L^\delta\partial^\alpha W_{ijk}\right)\\ &\quad -\nu\big(\Delta^2\partial^\alpha W_{ijk}, [\partial^\alpha, L^\delta]W_{ijk}\big) + \big(\partial^\alpha\mathcal{R}^\nu, [\partial^\alpha, L^\delta]W_{ijk}\big)\\ &\quad -\nu\big(\Delta^2\partial^\alpha W_{ijk}, \partial^\alpha JPJ\big((\mathcal{Q}^{ijk})^\delta - (\mathcal{Q}^{ijk})^a\big)\big)\\ &\quad +\big(\partial^\alpha\mathcal{R}^\nu, \partial^\alpha JPJ\big((\mathcal{Q}^{ijk})^\delta - (\mathcal{Q}^{ijk})^a\big)\big).\end{aligned}$$

In view of the estimates of Section 7.8, it suffices to estimate $\mathcal{D}^\nu$ uniformly in ν. Let us start with the estimate of the first term which will give the parabolic regularisation term. By using an integration by parts, we have

$$-\nu\big(\Delta^2\partial^\alpha W_{ijk}, L^\delta\partial^\alpha W_{ijk}\big) = -\nu\big(\Delta\partial^\alpha W_{ijk}, L^\delta\Delta\partial^\alpha W_{ijk}\big) - \nu\big(\Delta\partial^\alpha W_{ijk}, \big[\Delta, L^\delta\big]\partial^\alpha W_{ijk}\big). \tag{8.4}$$

By using estimates as in Proposition 7.28, we get the estimate

$$\begin{aligned}&-\nu\big(\Delta^2\partial^\alpha W_{ijk}, L^\delta\partial^\alpha W_{ijk}\big)\\ &\leq -\frac{\nu}{\overline{\omega}_{2,5}}\|\Delta\partial^\alpha W_{ijk}\|^2_{X^0} + \nu\|\Delta\partial^\alpha W_{ijk}\|^2_{L^2} + \nu\,\overline{\omega}_{l,S}\|\langle\nabla\rangle\partial^\alpha W_{ijk}\|_{X^0}\|\Delta W_{ijk}\|_{X^l},\end{aligned} \tag{8.5}$$

provided $l \geq 2$ and $S \geq 5$. Next, from standard interpolation in Sobolev spaces, we infer for every $\zeta > 0$

$$\nu\|\Delta\partial^\alpha W_{ijk}\|^2_{L^2} + \nu\,\overline{\omega}_{l,S}\|\langle\nabla\rangle\partial^\alpha W_{ijk}\|_{X^0}\|\Delta W_{ijk}\|_{X^l} \leq \zeta\,\nu\|\Delta W_{ijk}\|^2_{X^l} + C(\zeta)\,\overline{\omega}_{l,S}\|W_{ijk}\|^2_{X^l}$$

for some $C(\zeta) > 0$ independent of $\nu \in (0,1]$ and hence, we can choose ζ sufficiently small to get

$$\nu\|\Delta\partial^\alpha W_{ijk}\|^2_{L^2} + \nu\,\overline{\omega}_{l,S}\|\langle\nabla\rangle\partial^\alpha W_{ijk}\|_{X^0}\|\Delta W_{ijk}\|_{X^l} \leq \frac{\nu}{4\,\overline{\omega}_{2,5}}\|\Delta W_{ijk}\|^2_{X^l} + \overline{\omega}_{l,S}\|U\|^2_{l+3}. \tag{8.6}$$

This yields thanks to (8.5)

$$-\nu\big(\Delta^2\partial^\alpha W_{ijk}, L^\delta\partial^\alpha W_{ijk}\big) \leq -\frac{3\nu}{4\,\overline{\omega}_{2,5}}\|\Delta\partial^\alpha W_{ijk}\|^2_{X^0} + \overline{\omega}_{l,S}\|U\|^2_{l+3}. \tag{8.7}$$

Next, we can study the second term in the right hand-side of the expression defining $\mathcal{D}^\nu$. From the form (8.3) of $\mathcal{R}^\nu$, we first get

$$\begin{aligned}\big(\partial^\alpha\mathcal{R}^\nu, L^\delta\partial^\alpha W_{ijk}\big) &= \nu\Big(\partial^\alpha\big([Z^\delta, \Delta^2]W_{ijk}[1]\big), -\nabla\cdot\big(v^\delta\partial^\alpha W_{ijk}[1]\big) + \partial_x\partial^\alpha W_{ijk}[1]\Big)\\ &\quad +\nu\Big(\partial^\alpha\big([Z^\delta, \Delta^2]W_{ijk}[1]\big), G^\delta\partial^\alpha W_{ijk}[2]\Big)\\ &\equiv R_1^\nu + R_2^\nu.\end{aligned} \tag{8.8}$$

Let us start with the estimate of R_2^ν which is the most difficult term to handle. Expanding the commutator, we need to estimate

$$\nu\Big(\partial^\alpha\big(\nabla^\beta Z^\delta\nabla^\gamma W_{ijk}[1]\big), G^\delta\partial^\alpha W_{ijk}[2]\Big), \quad |\beta| + |\gamma| = 4, \quad |\gamma| \leq 3.$$

Using integrations by parts, we redistribute the 4 derivatives of ∇^β and ∇^γ equally to both sides of the above scalar product. In addition, we invoke Lemma 7.22 to deal with the Z^δ contribution. This leads to

$$\nu\Big|\Big(\partial^\alpha\big(\nabla^\beta Z^\delta\nabla^\gamma W_{ijk}[1]\big), G^\delta\partial^\alpha W_{ijk}[2]\Big)\Big| \leq \zeta\,\nu\|\Delta W_{ijk}\|^2_{X^l} + C(\zeta)\overline{\omega}_{l,S}\|U\|^2_{X^{l+3}}.$$

Coming back to (8.8), the term R_1^ν can be estimated in a similar way. Consequently, we find

$$\big|\big(\partial^\alpha\mathcal{R}^\nu, L^\delta\partial^\alpha W_{ijk}\big)\big| \leq \zeta\,\nu\,\|\Delta W_{ijk}\|^2_{X^l} + C(\zeta)\overline{\omega}_{l,S}\|U\|^2_{X^{l+3}}\,. \tag{8.9}$$

The third and the fourth terms in the expression defining $\mathcal{D}^\nu$ can be handled very similarly. Finally, the last two terms can be estimated by using estimates like in Proposition 7.27 (here one needs

to revisit the proof of Proposition 7.27 and to follow more carefully the dependence in $\overline{\omega}_{m,S}$). In summary, we get the following estimate

$$\mathcal{D}^\nu \leq -\frac{3\nu}{4\,\overline{\omega}_{2,5}}\|\Delta W_{ijk}\|^2_{X^l} + \zeta\,\nu\,\|\Delta W_{ijk}\|^2_{X^l} + C(\zeta)\overline{\omega}_{l,S}\|U\|^2_{l+3}. \tag{8.10}$$

From the energy identity, we can use Lemma 7.31, Lemma 7.34 and Lemma 7.35 as in Section 7.8 to get the energy estimate

$$E_l(t) + \nu\sum_{i,j,k}\int_0^t \|\Delta W_{ijk}(\tau)\|^2_{X^l}d\tau \leq \omega\Big(\|R^{ap}\|_{X_t^{l+3}} + \|V^a\|_{\mathcal{W}_t^{l+S}} + \|U\|_{X_t^{l+3}}\Big)$$
$$\times\Big(|E_l(0)| + \int_0^t\Big(\zeta\,\nu\sum_{i,j,k}\|\Delta W_{ijk}(\tau)\|^2_{X^l} + C(\zeta)\|U(\tau)\|^2_{X^{l+3}}\Big)d\tau\Big).$$

By choosing $\zeta < 1/2$, we thus get the uniform estimate

$$E_l(t) + \frac{\nu}{2}\sum_{i,j,k}\int_0^t \|\Delta W_{ijk}(\tau)\|^2_{X^l}d\tau$$
$$\leq \omega\big(\|R^{ap}\|_{X_t^{l+3}} + \|V^a\|_{\mathcal{W}_t^{l+S}} + \|U\|_{X_t^{l+3}}\big)\big(|E_l(0)| + \int_0^t \|U(\tau)\|^2_{X^{l+3}}d\tau\big).$$

In particular

$$E_l(t) \leq \omega\Big(\|R^{ap}\|_{X_t^{l+3}} + \|V^a\|_{\mathcal{W}_t^{l+S}} + \|U\|_{X_t^{l+3}}\Big)\Big(|E_l(0)| + \int_0^t \|U(\tau)\|^2_{X^{l+3}}d\tau\Big).$$

From Lemma 7.35 and a standard continuation argument, this yields that $\|U^\nu(t)\|_{X^{l+3}}$ is bounded uniformly in ν on an interval of time $[0,T]$ independent of ν. This allows to use strong compactness arguments in a classical way in order to prove that a subsequence of U^ν converges locally strongly in H^{s_0} to a solution of (7.2).

Acknowledgments. We benefited from discussions with David Lannes and Jean-Claude Saut on the water waves equation. We are also grateful to Francis Nier for useful discussion on the spectral theory issues of this paper.

References

[1] T. Alazard and G. Metivier, *Paralinearization of the Dirichlet to Neumann operator and regularity of three-dimensional water waves*, preprint 2009, arXiv:0901.2888v1.

[2] S. Alinhac, *Existence d'ondes de raréfaction pour des systèmes quasilinéaires hyperboliques multidimensionnels [Existence of rarefaction waves for multidimensional hyperbolic quasilinear systems].* Comm. Partial Differential Equations 14 (1989), 173-230.

[3] B. Alvarez-Samaniego, D. Lannes, *Large time existence for 3D water-waves and asymptotics*, Invent. Math. 171 (2008), 485-541.

[4] D. Ambrose and N. Masmoudi, The zero surface tension limit of two-dimensional water-waves. Comm. Pure Appl. Math. 58 (2005), 1287-1315.

[5] J.C Alexander, R.L. Pego, R.L. Sachs, *On the transverse instability of solitary waves in the Kadomtsev-Petviashvili equation*, Phys. Lett. A, 226 (1997), 187-192.

[6] C. Amick, K. Kirchgässner, *A theory of solitary water-waves in the presence of surface tension*, Arch. Ration. Mech. Analysis 105 (1989), 1–49.

[7] T. J. Bridges, *Transverse instability of solitary-wave states of the water-wave problem,* J. Fluid Mech. 439 (2001), 255–278.

[8] B. Buffoni, E.N. Dancer and J. F. Toland, *The regularity and local bifurcation of steady periodic water waves*, Arch. Rational Mech. Anal.152 (2000), 207-240.

[9] D. Christodoulou and H. Lindblad,*On the motion of the free surface of a liquid*, Comm. Pure Appl. Math. 53 (2000), 1536-1602.
[10] A. Constantin and W. Strauss. *Exact periodic traveling water-waves with vorticity* C. R. Math. Acad. Sci. Paris 335 (2002), 797-800.
[11] D. Coutand, and S. Shkoller, *Well-posedness of the free-surface incompressible Euler equations with or without surface tension*, J. Amer. Math. Soc., 20 (2007),829–930.
[12] W. Craig, *An existence theory for water waves and the Boussinesq and Korteweg-de Vries scaling limits*. Comm Partial Differential Equations 10 (1985), 787-1003.
[13] J. Dieudonné, *Calcul infinitésimal*, Collection Méthodes, Hermann, Paris, 1980.
[14] M. Fedoriuk, *Metod perevala*, Mir, Moscow, 1977 (in Russian).
[15] D. Gilbarg, N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Second Edition, Springer, 1998.
[16] E. Grenier, *On the nonlinear instability of Euler and Prandtl equations*, Comm. Pures Appl. Math. 53 (2000), 1067-1091.
[17] M. Grillakis, J. Shatah, and W. Strauss. *Stability theory of solitary waves in the presence of symmetry II.* J. Funct. Anal. 94 (1990), 308–348.
[18] M. Groves, M. Haragus, S.M. Sun, *Transverse instability of gravity-capillary line solitary waves*, C.R. Acad. Sci. Paris 333 (2001), 421-426.
[19] M. Groves and A. Mielke, *A spatial dynamics approach to three-dimensional gravity-capillary steady water waves* Proc. Roy. Soc. Edinb. A 131 (2001), 83-136.
[20] D. Henry, Geometric theory of semilinear parabolic equations. Lecture Notes in Mathematics, 840. Springer-Verlag, Berlin-New York, 1981. iv+348 pp. ISBN: 3-540-10557-3
[21] T. Iguchi, *A shallow water approximation for water waves*, Preprint 2008.
[22] G. Iooss and K. Kirchgässner. *Water waves for small surface tension: an approach via normal form*. Proc. Roy. Soc. Edinb. A 122 (1992), 267-299.
[23] G. Iooss and P. Plotnikov. *Small divisor problem in the theory of three-dimensional water gravity waves*. Mem. Amer. Math. Soc, to appear.
[24] D. Lannes, *Well-posedness of the water-waves equations*, Journal AMS 18 (2005), 605-654.
[25] H. Lindblad, *Well-posedness for the motion of an incompressible liquid with the free surface boundary*, Ann. of Math. (2) 162 (2005), 109-194.
[26] A. Mielke, *On the energetic stability of solitary water waves*, Phil. Trans. R. Soc. Lond. A 360 (2002), 2337-2358.
[27] Mei-Ming and Zhifei Zhang, Well-posedness of the water wave problem with surface tension, preprint 2008.
[28] R. Pego, S.M. Sun, *On the transverse linear instability of solitary water waves with large surface tension*, Proc. Royal Soc. Edinburgh 134 (2004), 733-752.
[29] R. Pego, M. Weinstein, *Eigenvalues, and instabilities of solitary waves*, Phil. Trans. R. Soc. London A 340 (1992), 47-97.
[30] P. I. Plotnikov and J. F. Toland, *Nash Moser theory for standing water waves*, Arch. Rational. Mech. Anal. 159 (2001), 1-83.
[31] F. Rousset, N. Tzvetkov, *Transverse nonlinear instability for two-dimensional dispersive models* , Ann. IHP, Analyse Non Linéaire, 26 (2009) 477-496.
[32] F. Rousset, N. Tzvetkov, *Transverse nonlinear instability for some Hamiltonian PDE's*, J. Math.Pures Appl. 90 (2008), 550-590.
[33] G. Schneider and C. E. Wayne, *The rigorous approximation of long wavelength capillariy gravity waves*, Arch. Rational. Mech. Anal. 162 (2002), 247-285.
[34] J. Shatah, C. Zeng, *Geometry and a priori estimates for free boundary problems of the Euler equation*, Comm. Pure Appl. Math., 61 (2008),698–744.
[35] M. Taylor, *Partial Differential Equations II*, Springer 1997.
[36] S. Wu, *Well-posedness in Sobolev spaces of the full water wave problem in 3-D*, J. Amer. Math. Soc.,12 (1999),445–495.
[37] V. Zakharov, *Stability of periodic waves of finite amplitude on the surface of a deep fluid*, J. Appl. Mech. Tech. Phys. 9 (1968), 190-194.
[38] V. Zakharov, *Instability and nonlinear oscillations of solitons*, JEPT Lett. 22 (1975), 172-173.

IRMAR, Université de Rennes 1, campus de Beaulieu, 35042 Rennes cedex, France
E-mail address: frederic.rousset@univ-rennes1.fr

Département de Mathématiques, Université Lille I, 59 655 Villeneuve d'Ascq Cedex, France
E-mail address: nikolay.tzvetkov@math.univ-lille1.fr